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SOME PROPERTIES OF APPROXIMANTS FOR BRANCHED CONTINUED FRACTIONS OF THE SPECIAL FORM WITH POSITIVE AND ALTERNATING-SIGN PARTIAL NUMERATORS

The paper deals with research of convergence for one of the generalizations of continued fractions — branched continued fractions of the special form with two branches. Such branched continued fractions, similarly as the two-dimensional continued fractions and the branched continued fractions with two independent variables are connected with the problem of the correspondence between a formal double power series and a sequence of the rational approximants of a function of two variables.

Unlike continued fractions, approximants of which are constructed unambiguously, there are many ways to construct approximants of branched continued fractions of the general and the special form. The paper examines the ordinary approximants and one of the structures of figured approximants of the studied branched continued fractions, which is connected with the problem of correspondence.

We consider some properties of approximants of such fractions, whose partial numerators are positive and alternating-sign and partial denominators are equal to one. Some necessary and sufficient conditions for figured convergence are established. It is proved that under these conditions from the convergence of the sequence of figured approximants it follows the convergence of the sequence of ordinary approximants to the same limit.

Key words and phrases: branched continued fraction of the special form, ordinary approximants, figured approximants, convergence, figured convergence.

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INTRODUCTION

The paper is devoted to study of the branched continued fractions (BCF) of the form

$$b_0 + F_{0,0} + \prod_{i=1}^{\infty} \frac{a_{i,0}}{1 + F_{i,0}} + \prod_{i=1}^{\infty} \frac{a_{0,i}}{1 + F_{0,i}}, \quad (1)$$

where $F_{i,j}$ are continued fractions (CF)

$$F_{i,j} = \prod_{p=1}^{\infty} \frac{a_{p+i,p+j}}{1} = 1 + \frac{a_{1+i,1+j}}{1 + \frac{a_{2+i,2+j}}{1 + \dots}}, \quad i = 0, 1, \dots, \quad j = 0, 1, \dots, \quad (2)$$

$b_0, a_{k,j}, j = 0, 1, \dots, k = 0, 1, \dots, k + j \geq 1$, are complex numbers or functions of two variables which are defined in some set $D \subset \mathbb{C}^2$.

If all elements of BCF (1)–(2) are numbers, then BCF (1)–(2) is said to be numerical BCF of the special form. If some or all elements of BCF (1)–(2) are functions, then BCF (1)–(2) is said to be functional BCF of the special form.

The above mentioned BCF is one of two-dimensional generalizations of continued fractions, which were offered for the solution of correspondence problem between a formal double power series (FDPS) and a sequence of the rational approximants of a function of two variables [9, 12, 13]. Functional BCF of the special form is corresponding to FDPS

$$\sum_{i+j \geq 0}^{\infty} c_{i,j} z_1^i z_2^j, \quad (3)$$

if the expansion of its n th approximant into FDPS $\sum_{i+j \geq 0}^{\infty} c_{i,j}^{(n)} z_1^i z_2^j$ coincides with the FDPS (3) to all terms of power n inclusively, that is $c_{i,j}^{(n)} = c_{i,j}, i + j \leq n$.

In the paper [13] it is shown, that BCF (1)–(2) is corresponding to the FDPS (3), if $b_0 = c_{0,0}, a_{i,0} = b_{i,0} z_1, a_{0,i} = b_{0,i} z_2, a_{i,j} = b_{i,j} z_1 z_2, i, j = 1, 2, \dots$, the coefficients $b_{k,0}, b_{0,k}, a_{k,j}, j, k = 1, 2, \dots$, are calculating by formulas in term of the coefficients of the FDPS (3), and the n th approximants \tilde{f}_n are defined as follows

$$\tilde{f}_0 = b_0, \tilde{f}_n = b_0 + F_{0,0}^{\left(\left[\frac{n}{2}\right]\right)} + \prod_{k=1}^n \frac{a_{i,0}}{1 + F_{i,0}^{\left(\left[\frac{n-i}{2}\right]\right)}} + \prod_{k=1}^n \frac{a_{0,i}}{1 + F_{0,i}^{\left(\left[\frac{n-i}{2}\right]\right)}}, \quad n = 1, 2, \dots, \quad (4)$$

where $[\alpha]$ is an integer part of a real number α ,

$$F_{i,j}^{(0)} = 0, \quad F_{i,j}^{(k)} = \prod_{p=1}^k \frac{a_{p+i,p+j}}{1}, \quad i, j = 0, 1, \dots, \quad k = 1, 2, \dots \quad (5)$$

Finite continued fractions (5) are called the k th approximants of CF (2).

We can construct the approximants of BCF in different ways. Ordinary n th approximants of BCF (1)–(2) are defined as follows

$$f_0 = b_0, f_n = b_0 + F_{0,0}^{(n)} + \prod_{k=1}^n \frac{a_{i,0}}{1 + F_{i,0}^{(n-i)}} + \prod_{k=1}^n \frac{a_{0,i}}{1 + F_{0,i}^{(n-i)}}, \quad n = 1, 2, \dots$$

Approximants \tilde{f}_n from (4) are examples of so called figured approximants [5]. Expressions

$$Q_{i,0}^{(0)} = 1, Q_{i,0}^{(k+1)} = 1 + F_{i,0}^{(k+1)} + \frac{a_{i+1,0}}{Q_{i+1,0}^{(k)}}, \quad i = 1, 2, \dots, \quad k = 0, 1, \dots, \quad (6)$$

$$Q_{0,i}^{(0)} = 1, Q_{0,i}^{(k+1)} = 1 + F_{0,i}^{(k+1)} + \frac{a_{0,i+1}}{Q_{0,i+1}^{(k)}}, \quad i = 1, 2, \dots, \quad k = 0, 1, \dots, \quad (7)$$

are said to be the tails of ordinary approximants for BCF (1)–(2). Tails of figured approximants (4) for BCF (1)–(2) are defined by following formulas

$$\tilde{Q}_{i,0}^{(0)} = 1, \tilde{Q}_{i,0}^{(k+1)} = 1 + F_{i,0}^{\left(\left[\frac{k+1}{2}\right]\right)} + \frac{a_{i+1,0}}{\tilde{Q}_{i+1,0}^{(k)}}, \quad i = 1, 2, \dots, \quad k = 0, 1, \dots, \quad (8)$$

$$\tilde{Q}_{0,i}^{(0)} = 1, \tilde{Q}_{0,i}^{(k+1)} = 1 + F_{0,i}^{\left(\left[\frac{k+1}{2}\right]\right)} + \frac{a_{0,i+1}}{\tilde{Q}_{0,i+1}^{(k)}}, \quad i = 1, 2, \dots, \quad k = 0, 1, \dots \quad (9)$$

Taking into account notations (6)–(9), it is possible to write

$$f_n = b_0 + F_{0,0}^{(n)} + \frac{a_{1,0}}{Q_{1,0}^{(n-1)}} + \frac{a_{0,1}}{Q_{0,1}^{(n-1)}}, \quad n = 1, 2, \dots,$$

$$\tilde{f}_n = b_0 + F_{0,0}^{\left(\left[\frac{n}{2}\right]\right)} + \frac{a_{1,0}}{\tilde{Q}_{1,0}^{(n-1)}} + \frac{a_{0,1}}{\tilde{Q}_{0,1}^{(n-1)}}, \quad n = 1, 2, \dots$$

Approximants f_k, \tilde{f}_k have sense if in process of reduction of BCF (calculations of their tails by formulas (6)–(9)) uncertainty of the type $\frac{0}{0}$ don't appears (it is assumed, that $\frac{1}{0} = \infty$, $\frac{1}{\infty} = 0$ and $\frac{\alpha_1}{0} + \dots + \frac{\alpha_m}{0} = \frac{0}{0}$, if $m > 1$). We say that BCF (1)–(2) is figured convergent if, beginning from some number n_0 , all its figured approximants have sense and there is finite limit $\tilde{f} = \lim_{n \rightarrow \infty} \tilde{f}_n$. The value of this limit can be the value of figured convergent BCF.

BCF (1)–(2) is said to be convergent, if beginning from some number n_0 , all its ordinary approximants have sense and there is finite limit $f = \lim_{n \rightarrow \infty} f_n$. The value of this limit also it is possible to assume as the value of convergent BCF.

BCF (1)–(2) is said to be divergent (figured divergent) if infinite numbers of its approximants (figured approximants) have not sense or there isn't only one finite limit of sequence of its approximants (figured approximants).

A lot of works of analytic theory of multidimensional generalization for continued fractions are devoted to research of convergence [5, 10]. This problem is important till now [3, 4, 6, 7].

For research of properties of sequences of approximants for BCF of the special form the formulas of difference for two their approximants are used. There are such formulas [5]:

$$\begin{aligned} \tilde{f}_n - \tilde{f}_m = & F_{0,0}^{\left(\left[\frac{n}{2}\right]\right)} - F_{0,0}^{\left(\left[\frac{m}{2}\right]\right)} + \sum_{i=1}^m \frac{(-1)^i \left(F_{i,0}^{\left(\left[\frac{n-i}{2}\right]\right)} - F_{i,0}^{\left(\left[\frac{m-i}{2}\right]\right)} \right) \prod_{j=1}^i a_{j,0}}{\prod_{j=1}^i \tilde{Q}_{j,0}^{(n-j)} \tilde{Q}_{j,0}^{(m-j)}} + \frac{(-1)^m \prod_{j=1}^{m+1} a_{j,0}}{\prod_{j=1}^{m+1} \tilde{Q}_{j,0}^{(n-j)} \prod_{j=1}^m \tilde{Q}_{j,0}^{(m-j)}} \\ & + \sum_{i=1}^m \frac{(-1)^i \left(F_{0,i}^{\left(\left[\frac{n-i}{2}\right]\right)} - F_{0,i}^{\left(\left[\frac{m-i}{2}\right]\right)} \right) \prod_{j=1}^i a_{0,j}}{\prod_{j=1}^i \tilde{Q}_{0,j}^{(n-j)} \tilde{Q}_{0,j}^{(m-j)}} + \frac{(-1)^m \prod_{j=1}^{m+1} a_{0,j}}{\prod_{j=1}^{m+1} \tilde{Q}_{0,j}^{(n-j)} \prod_{j=1}^m \tilde{Q}_{0,j}^{(m-j)}}, \quad n > m, \text{ and} \quad (10) \end{aligned}$$

$$\begin{aligned} f_n - \tilde{f}_m = & F_{0,0}^{(n)} - F_{0,0}^{\left(\left[\frac{m}{2}\right]\right)} + \sum_{i=1}^m \frac{(-1)^i \left(F_{i,0}^{(n-i)} - F_{i,0}^{\left(\left[\frac{m-i}{2}\right]\right)} \right) \prod_{j=1}^i a_{j,0}}{\prod_{j=1}^i Q_{j,0}^{(n-j)} \tilde{Q}_{j,0}^{(m-j)}} + \frac{(-1)^m \prod_{j=1}^{m+1} a_{j,0}}{\prod_{j=1}^{m+1} Q_{j,0}^{(n-j)} \prod_{j=1}^m \tilde{Q}_{j,0}^{(m-j)}} \\ & + \sum_{i=1}^m \frac{(-1)^i \left(F_{0,i}^{(n-i)} - F_{0,i}^{\left(\left[\frac{m-i}{2}\right]\right)} \right) \prod_{j=1}^i a_{0,j}}{\prod_{j=1}^i Q_{0,j}^{(n-j)} \tilde{Q}_{0,j}^{(m-j)}} + \frac{(-1)^m \prod_{j=1}^{m+1} a_{0,j}}{\prod_{j=1}^{m+1} Q_{0,j}^{(n-j)} \prod_{j=1}^m \tilde{Q}_{0,j}^{(m-j)}}, \quad n > m. \quad (11) \end{aligned}$$

We note that the formulas (10)–(11) have been established in assumption, that the values of all tails $\tilde{Q}_{0,k}^{(p)}, \tilde{Q}_{k,0}^{(p)}, Q_{0,k}^{(p)}, Q_{k,0}^{(p)}$, which appear in these formulas, differ from 0.

MAIN RESULTS

This work is the continuation of the study of properties of approximants for numerical BCF (1)–(2) with real elements [1,2]. We will consider BCF which elements satisfy such conditions

$$a_{i,j} > 0, |a_{i,0}| = (-1)^{i-1} a_{i,0} \neq 0, |a_{0,i}| = (-1)^{i-1} a_{0,i} \neq 0, i, j = 1, 2, \dots \quad (12)$$

Under the conditions (12) $F_{0,0}, F_{i,0}, F_{0,i}$ are the continued fractions with positive elements. It is well known [8, 11] that approximants of even order for such fractions generate a monotone increasing sequence, approximants of odd order for such fractions generate a monotone decreasing sequence and all approximants of even order are less than every approximants of odd order. Taking these results into account, we have

$$F_{0,0}^{(2m)} < F_{0,0}^{(2m+2)} < F_{0,0}^{(2n+1)} < F_{0,0}^{(2n-1)}, \quad m = 0, 1, \dots, \quad n = 1, 2, \dots, \quad (13)$$

$$F_{i,0}^{(2m)} < F_{i,0}^{(2m+2)} < F_{i,0}^{(2n+1)} < F_{i,0}^{(2n-1)}, \quad m = 0, 1, \dots, \quad n = 1, 2, \dots, \quad (14)$$

$$F_{0,i}^{(2m)} < F_{0,i}^{(2m+2)} < F_{0,i}^{(2n+1)} < F_{0,i}^{(2n-1)}, \quad m = 0, 1, \dots, \quad n = 1, 2, \dots \quad (15)$$

Theorem 1. *Let the elements of BCF (1)–(2) satisfy the conditions (12) and*

$$1 + a_{2i,0} > 0, \quad 1 + a_{0,2i} > 0, \quad i = 1, 2, \dots \quad (16)$$

Then the following inequalities are true

$$\tilde{f}_{4m} < \tilde{f}_{4m+4} < \tilde{f}_{4n+2} < \tilde{f}_{4n-2}, \quad m = 0, 1, \dots, \quad n = 1, 2, \dots, \quad (17)$$

$$\tilde{f}_{4m} < \tilde{f}_{4m+2p+3} < \tilde{f}_{4m+2}, \quad m = 0, 1, \dots, \quad p = 1, 2, \dots, \quad (18)$$

and the sequences $\{\tilde{f}_{4p}\}, \{\tilde{f}_{4p+2}\}, p = 0, 1, \dots$, converge.

Proof. Let k be an arbitrary natural number. Using definitions (8)–(9), by induction on p let us show that the following inequalities are valid

$$1 \leq \tilde{Q}_{2k,0}^{(p)} \leq 1 + a_{2k+1,1} + \frac{a_{2k+1,0}}{1 - |a_{2k+2,0}|}, \quad p = 0, 1, \dots, \quad (19)$$

$$1 - |a_{2k,0}| \leq \tilde{Q}_{2k-1,0}^{(p)} \leq 1 + a_{2k,1}, \quad p = 0, 1, \dots \quad (20)$$

Indeed, for $p = 0$ and $p = 1$ we have

$$\tilde{Q}_{2k-1,0}^{(0)} = \tilde{Q}_{2k,0}^{(0)} = 1, \quad 0 < 1 - |a_{2k,0}| = \tilde{Q}_{2k-1,0}^{(1)} < 1, \quad \tilde{Q}_{2k,0}^{(1)} = 1 + a_{2k+1,0}, \quad k = 1, 2, \dots$$

In assumption that inequality (19) is true for $p = r$, we obtain

$$0 < 1 - |a_{2k,0}| < 1 + F_{2k-1,0}^{(\lfloor \frac{p+1}{2} \rfloor)} - \frac{|a_{2k,0}|}{1} \leq \tilde{Q}_{2k-1,0}^{(p+1)} = 1 + F_{2k-1,0}^{(\lfloor \frac{p+1}{2} \rfloor)} - \frac{|a_{2k,0}|}{\tilde{Q}_{2k,0}^{(p)}} < 1 + a_{2k,1},$$

i.e. (20) is valid for $p = r + 1$. Assuming that inequality (20) holds true for $p = r$ we get

$$1 < 1 + \frac{a_{2k+1,0}}{1 + a_{2k+2,1}} < \tilde{Q}_{2k,0}^{(p+1)} = 1 + F_{2k,0}^{(\lfloor \frac{p+1}{2} \rfloor)} + \frac{a_{2k+1,0}}{\tilde{Q}_{2k+1,0}^{(p)}} \leq 1 + a_{2k+1,1} + \frac{a_{2k+1,0}}{1 - |a_{2k+2,0}|},$$

i.e. inequality (19) is valid for $p = r + 1$. Hence, estimations (19)–(20) are true for arbitrary k, p . Analogously we verify validity of inequalities

$$1 \leq \tilde{Q}_{0,2k}^{(p)} \leq 1 + a_{1,2k+1} + \frac{a_{0,2k+1}}{1 - |a_{0,2k+2}|}, \quad k = 1, 2, \dots, \quad p = 0, 1, \dots, \quad (21)$$

$$1 - |a_{0,2k}| \leq \tilde{Q}_{0,2k-1}^{(p)} \leq 1 + a_{1,2k}, \quad k = 1, 2, \dots, \quad p = 0, 1, \dots \quad (22)$$

Further we consider the differences $\tilde{f}_{4m+2p+l} - \tilde{f}_{4m}$, $\tilde{f}_{4n+2p+l-2} - \tilde{f}_{4n-2}$, $m, n, p = 1, 2, \dots$, $l = 0, 1, \dots$, using the formula (10). Let

$$\tilde{Z}_{n,0}^{(1)} = 0, \quad \tilde{Z}_{n,m}^{(1)} = \sum_{i=1}^m \frac{(-1)^i \left(F_{i,0}^{\left(\left[\frac{n-i}{2}\right]\right)} - F_{i,0}^{\left(\left[\frac{m-i}{2}\right]\right)} \right) \prod_{j=1}^i a_{j,0}}{\prod_{j=1}^i \tilde{Q}_{j,0}^{(n-j)} \tilde{Q}_{j,0}^{(m-j)}}, \quad n > m;$$

$$\tilde{Z}_{n,0}^{(2)} = 0, \quad \tilde{Z}_{n,m}^{(2)} = \sum_{i=1}^m \frac{(-1)^i \left(F_{0,i}^{\left(\left[\frac{n-i}{2}\right]\right)} - F_{0,i}^{\left(\left[\frac{m-i}{2}\right]\right)} \right) \prod_{j=1}^i a_{0,j}}{\prod_{j=1}^i \tilde{Q}_{0,j}^{(n-j)} \tilde{Q}_{0,j}^{(m-j)}}, \quad n > m.$$

Then

$$\begin{aligned} \tilde{f}_{4m+2p+l} - \tilde{f}_{4m} &= F_{0,0}^{(2m+p+\lceil \frac{l}{2} \rceil)} - F_{0,0}^{(2m)} + \tilde{Z}_{4m+2p+l,4m}^{(1)} + \tilde{Z}_{4m+2p+l,4m}^{(2)} \\ &\quad + \frac{\prod_{j=1}^{4m+1} a_{j,0}}{\prod_{j=1}^{4m+1} \tilde{Q}_{j,0}^{(4m+2p+l-j)} \prod_{j=1}^{4m} \tilde{Q}_{j,0}^{(4m-j)}} + \frac{\prod_{j=1}^{4m+1} a_{0,j}}{\prod_{j=1}^{4m+1} \tilde{Q}_{0,j}^{(4m+2p+l-j)} \prod_{j=1}^{4m} \tilde{Q}_{0,j}^{(4m-j)}}. \end{aligned}$$

Taking into account conditions (12) and inequalities (13)–(15), (19)–(22), we have

$$\prod_{j=1}^{4m+1} a_{j,0} = \prod_{j=1}^{4m+1} |a_{j,0}| > 0, \quad \prod_{j=1}^{4m+1} a_{0,j} = \prod_{j=1}^{4m+1} |a_{0,j}| > 0, \quad (23)$$

$$\begin{aligned} \tilde{Z}_{4m+2p+l,4m}^{(1)} &= - \sum_{i=1}^m \frac{\left(F_{4i-3,0}^{(2m-2i+p+1+\lceil \frac{l+1}{2} \rceil)} - F_{4i-3,0}^{(2m-2i+1)} \right) \prod_{j=1}^{4i-3} |a_{j,0}|}{\prod_{j=1}^{4i-3} \tilde{Q}_{j,0}^{(4m+2p+l-j)} \tilde{Q}_{j,0}^{(4m-j)}} \\ &\quad - \sum_{i=1}^m \frac{\left(F_{4i-2,0}^{(2m-2i+p+1+\lceil \frac{l}{2} \rceil)} - F_{4i-2,0}^{(2m-2i+1)} \right) \prod_{j=1}^{4i-2} |a_{j,0}|}{\prod_{j=1}^{4i-2} \tilde{Q}_{j,0}^{(4m+2p+l-j)} \tilde{Q}_{j,0}^{(4m-j)}} + \sum_{i=1}^m \frac{\left(F_{4i-1,0}^{(2m-2i+p+\lceil \frac{l+1}{2} \rceil)} - F_{4i-1,0}^{(2m-2i)} \right) \prod_{j=1}^{4i-1} |a_{j,0}|}{\prod_{j=1}^{4i-1} \tilde{Q}_{j,0}^{(4m+2p+l-j)} \tilde{Q}_{j,0}^{(4m-j)}} \\ &\quad + \sum_{i=1}^m \frac{\left(F_{4i,0}^{(2m-2i+p+\lceil \frac{l}{2} \rceil)} - F_{4i,0}^{(2m-2i)} \right) \prod_{j=1}^{4i} |a_{j,0}|}{\prod_{j=1}^{4i} \tilde{Q}_{j,0}^{(4m+2p+l-j)} \tilde{Q}_{j,0}^{(4m-j)}} = \sum_{i=1}^m \frac{\left| F_{4i-3,0}^{(2m-2i+p+1+\lceil \frac{l+1}{2} \rceil)} - F_{4i-3,0}^{(2m-2i+1)} \right| \prod_{j=1}^{4i-3} |a_{j,0}|}{\prod_{j=1}^{4i-3} \tilde{Q}_{j,0}^{(4m+2p+l-j)} \tilde{Q}_{j,0}^{(4m-j)}} \end{aligned}$$

$$\begin{aligned}
& + \sum_{i=1}^m \frac{\left| F_{4i-2,0}^{(2m-2i+p+1+\lfloor \frac{l}{2} \rfloor)} - F_{4i-2,0}^{(2m-2i+1)} \right| \prod_{j=1}^{4i-2} |a_{j,0}|}{\prod_{j=1}^{4i-2} \tilde{Q}_{j,0}^{(4m+2p+l-j)} \tilde{Q}_{j,0}^{(4m-j)}} + \sum_{i=1}^m \frac{\left| F_{4i-1,0}^{(2m-2i+p+\lfloor \frac{l+1}{2} \rfloor)} - F_{4i-1,0}^{(2m-2i)} \right| \prod_{j=1}^{4i-1} |a_{j,0}|}{\prod_{j=1}^{4i-1} \tilde{Q}_{j,0}^{(4m+2p+l-j)} \tilde{Q}_{j,0}^{(4m-j)}} \\
& + \sum_{i=1}^m \frac{\left| F_{4i,0}^{(2m-2i+p+\lfloor \frac{l}{2} \rfloor)} - F_{4i,0}^{(2m-2i)} \right| \prod_{j=1}^{4i} |a_{j,0}|}{\prod_{j=1}^{4i} \tilde{Q}_{j,0}^{(4m+2p+l-j)} \tilde{Q}_{j,0}^{(4m-j)}} > 0.
\end{aligned}$$

Similarly $\tilde{Z}_{4m+2p+l,4m}^{(2)} > 0$. Consequently

$$\tilde{f}_{4m+2p+l} - \tilde{f}_{4m} > 0, \quad m, l = 0, 1, \dots, \quad p = 1, 2, \dots, \quad (24)$$

$$\begin{aligned}
\tilde{f}_{4n+2p+l-2} - \tilde{f}_{4n-2} &= F_{0,0}^{(2n+p-1+\lfloor \frac{l}{2} \rfloor)} - F_{0,0}^{(2n-1)} + \tilde{Z}_{4n+2p+l-2,4n-2}^{(1)} + \tilde{Z}_{4n+2p+l-2,4n-2}^{(2)} \\
&+ \frac{\prod_{j=1}^{4n-1} a_{j,0}}{\prod_{j=1}^{4n-1} \tilde{Q}_{j,0}^{(4n+2p+l-2-j)} \prod_{j=1}^{4n-2} \tilde{Q}_{j,0}^{(4n-2-j)}} + \frac{\prod_{j=1}^{4n-1} a_{0,j}}{\prod_{j=1}^{4n-1} \tilde{Q}_{0,j}^{(4n+2p+l-2-j)} \prod_{j=1}^{4n-2} \tilde{Q}_{0,j}^{(4n-2-j)}}, \quad (25)
\end{aligned}$$

$$\prod_{j=1}^{4n-1} a_{j,0} = - \prod_{j=1}^{4n-1} |a_{j,0}| < 0, \quad \prod_{j=1}^{4n-1} a_{0,j} = - \prod_{j=1}^{4n-1} |a_{0,j}| < 0,$$

$$\begin{aligned}
\tilde{Z}_{4n+2p+l-2,4n-2}^{(1)} &= - \sum_{i=1}^n \frac{\left(F_{4i-3,0}^{(2n-2i+p+\lfloor \frac{l+1}{2} \rfloor)} - F_{4i-3,0}^{(2n-2i)} \right) \prod_{j=1}^{4i-3} |a_{j,0}|}{\prod_{j=1}^{4i-3} \tilde{Q}_{j,0}^{(4n+2p+l-2-j)} \tilde{Q}_{j,0}^{(4n-2-j)}} \\
&+ \sum_{i=1}^{n-1} \frac{\left(F_{4i-1,0}^{(2n+p-2i-1+\lfloor \frac{l+1}{2} \rfloor)} - F_{4i-1,0}^{(2n-2i-1)} \right) \prod_{j=1}^{4i-1} |a_{j,0}|}{\prod_{j=1}^{4i-1} \tilde{Q}_{j,0}^{(4n+2p+l-2-j)} \tilde{Q}_{j,0}^{(4n-2-j)}} - \sum_{i=1}^n \frac{\left(F_{4i-2,0}^{(2n-2i+p+\lfloor \frac{l}{2} \rfloor)} - F_{4i-2,0}^{(2n-2i)} \right) \prod_{j=1}^{4i-2} |a_{j,0}|}{\prod_{j=1}^{4i-2} \tilde{Q}_{j,0}^{(4n+2p+l-2-j)} \tilde{Q}_{j,0}^{(4n-2-j)}} \\
&+ \sum_{i=1}^{n-1} \frac{\left(F_{4i,0}^{(2n+p-2i-1+\lfloor \frac{l}{2} \rfloor)} - F_{4i,0}^{(2n-2i-1)} \right) \prod_{j=1}^{4i} |a_{j,0}|}{\prod_{j=1}^{4i} \tilde{Q}_{j,0}^{(4n+2p+l-2-j)} \tilde{Q}_{j,0}^{(4n-2-j)}} = - \sum_{i=1}^n \frac{\left| F_{4i-3,0}^{(2n-2i+p+\lfloor \frac{l+1}{2} \rfloor)} - F_{4i-3,0}^{(2n-2i)} \right| \prod_{j=1}^{4i-3} |a_{j,0}|}{\prod_{j=1}^{4i-3} \tilde{Q}_{j,0}^{(4n+2p+l-2-j)} \tilde{Q}_{j,0}^{(4n-j-2)}} \\
&- \sum_{i=1}^n \frac{\left| F_{4i-2,0}^{(2n-2i+p+\lfloor \frac{l}{2} \rfloor)} - F_{4i-2,0}^{(2n-2i)} \right| \prod_{j=1}^{4i-2} |a_{j,0}|}{\prod_{j=1}^{4i-2} \tilde{Q}_{j,0}^{(4n+2p+l-j-2)} \tilde{Q}_{j,0}^{(4n-j-2)}} - \sum_{i=1}^{n-1} \frac{\left| F_{4i-1,0}^{(2n-2i+p-1+\lfloor \frac{l+1}{2} \rfloor)} - F_{4i-1,0}^{(2n-2i-1)} \right| \prod_{j=1}^{4i-1} |a_{j,0}|}{\prod_{j=1}^{4i-1} \tilde{Q}_{j,0}^{(4n+2p+l-j-2)} \tilde{Q}_{j,0}^{(4n-j-2)}} \\
&- \sum_{i=1}^{n-1} \frac{\left| F_{4i,0}^{(2n-2i+p-1+\lfloor \frac{l}{2} \rfloor)} - F_{4i,0}^{(2n-2i-1)} \right| \prod_{j=1}^{4i} |a_{j,0}|}{\prod_{j=1}^{4i} \tilde{Q}_{j,0}^{(4n+2p+l-j-2)} \tilde{Q}_{j,0}^{(4n-j-2)}} < 0, \quad \tilde{Z}_{4n+2p+l-2,4n-2}^{(2)} < 0.
\end{aligned}$$

Consequently

$$\tilde{f}_{4n+2p+l-2} - \tilde{f}_{4n-2} < 0, \quad l = 0, 1, \dots, \quad n, p = 1, 2, \dots \quad (26)$$

From the inequalities (24), (26), where $l = 0$, it follows “fork” property for figured approximants of even order. This property is described by system of inequalities (17). Therefore sequences $\{\tilde{f}_{4k}\}$, $\{\tilde{f}_{4k+2}\}$ are monotone, bounded and convergent. From (24), (26), where $l = 3$, $n = m + 1$, we obtain inequality (18). \square

Proposition. *BCF (1)–(2), with elements that satisfy conditions (12) and (16), is figured convergent if and only if $\lim_{n \rightarrow \infty} (\tilde{f}_{4n+2} - \tilde{f}_{4n}) = 0$.*

Proof. It is above mentioned that under conditions (12) and (16) the sequences $\{\tilde{f}_{4k}\}$, $\{\tilde{f}_{4k+2}\}$ converge to finite limits. Condition $\lim_{n \rightarrow \infty} (\tilde{f}_{4n+2} - \tilde{f}_{4n}) = 0$ implies equality of these limits, i.e. convergence of the sequence $\{\tilde{f}_{2k}\}$. Taking into account inequality (18), we conclude that $\lim_{k \rightarrow \infty} \tilde{f}_{2k} = \lim_{k \rightarrow \infty} \tilde{f}_{2k-1}$, i.e. BCF (1)–(2) is figured convergent. \square

Theorem 2. *Divergence of the series*

$$\sum_{n=1}^{\infty} \prod_{k=1}^n (a_{k,k})^{(-1)^{n-k+1}}, \quad \sum_{n=1}^{\infty} \prod_{k=1}^n (a_{k+i,k})^{(-1)^{n-k+1}}, \quad \sum_{n=1}^{\infty} \prod_{k=1}^n (a_{k,k+j})^{(-1)^{n-k+1}},$$

$i, j = 1, 2, \dots$, is necessary condition of figured convergence of BCF (1)–(2) whose elements satisfy conditions (12) and (16).

Proof. Using well known results of analytic theory of continued fractions [8, 11], we conclude that continued fraction (2) converges if and only if the series $\sum_{n=1}^{\infty} \prod_{k=1}^n (a_{k+i,k+j})^{(-1)^{n-k+1}}$ diverges. It was shown above that every summand which appears in expressions for $\tilde{f}_{4m+2p+l} - \tilde{f}_{4m}$, $m, l = 0, 1, \dots, p = 1, 2, \dots$, is positive under conditions (12), (16). If series $\sum_{n=1}^{\infty} \prod_{k=1}^n (a_{k,k})^{(-1)^{n-k+1}}$ converges, then $\lim_{m \rightarrow \infty} (\tilde{f}_{4m+2} - \tilde{f}_{4m}) \geq \lim_{m \rightarrow \infty} (F_{0,0}^{(2m+1)} - F_{0,0}^{(2m)}) > 0$, i.e. BCF (1)–(2) diverges. Let there exists such i that series $\sum_{n=1}^{\infty} \prod_{k=1}^n (a_{k+i,k})^{(-1)^{n-k+1}}$ converges. Then taking into account inequalities (19), (20), we obtain

$$\begin{aligned} \tilde{f}_{4m+2} - \tilde{f}_{4m} &> \frac{\left| F_{i,0}^{(2m+1 - [\frac{i+1}{2}])} - F_{i,0}^{(2m - [\frac{i+1}{2}])} \right| \prod_{j=1}^i |a_{j,0}|}{\prod_{j=1}^i \tilde{Q}_{j,0}^{(4m+2-j)} \tilde{Q}_{j,0}^{(4m-j)}} \\ &= \frac{\left| F_{i,0}^{(2m+1 - [\frac{i+1}{2}])} - F_{i,0}^{(2m - [\frac{i+1}{2}])} \right| \prod_{j=1}^i |a_{j,0}|}{\prod_{j=1}^{[\frac{i+1}{2}]} \tilde{Q}_{2j-1,0}^{(4m-2j+3)} \tilde{Q}_{2j-1,0}^{(4m-2j+1)} \prod_{j=1}^{[\frac{i}{2}]} \tilde{Q}_{2j,0}^{(4m-2j+2)} \tilde{Q}_{2j,0}^{(4m-2j)}}, \quad 4m \geq i, \end{aligned}$$

$$\lim_{m \rightarrow \infty} (\tilde{f}_{4m+2} - \tilde{f}_{4m}) \geq \frac{\prod_{j=1}^i |a_{j,0}| \lim_{m \rightarrow \infty} \left| F_{i,0}^{(2m+1 - [\frac{i+1}{2}])} - F_{i,0}^{(2m - [\frac{i+1}{2}])} \right|}{\prod_{j=1}^{[\frac{i+1}{2}]} (1 + a_{2j,1})^2 \prod_{j=1}^{[\frac{i}{2}]} \left(1 + a_{2j+1,1} + \frac{a_{2j+1,0}}{1 - |a_{2j+2,0}|} \right)^2} > 0.$$

From assumption about convergence of the series $\sum_{n=1}^{\infty} \prod_{k=1}^n (a_{k,k+i})^{(-1)^{n-k+1}}$ for some value i and from inequalities (21), (22) it follows that

$$\lim_{m \rightarrow \infty} (\tilde{f}_{4m+2} - \tilde{f}_{4m}) \geq \frac{\prod_{j=1}^i |a_{0,j}| \lim_{m \rightarrow \infty} \left| F_{0,i}^{(2m+1 - [\frac{i+1}{2}])} - F_{0,i}^{(2m - [\frac{i+1}{2}])} \right|}{\prod_{j=1}^{[\frac{i+1}{2}]} (1 + a_{1,2j})^2 \prod_{j=1}^{[\frac{i}{2}]} \left(1 + a_{1,2j+1} + \frac{a_{0,2j+1}}{1 - |a_{0,2j+2}|} \right)^2} > 0.$$

□

Remark. “Fork property” for ordinary approximants of even order is not valid. Really, let $b_0 = 1, a_{i,k} = 1, a_{2k-1,0} = a_{0,2k-1} = 1, a_{2k,0} = a_{0,2k} = -\frac{1}{2}, i, k = 1, 2, \dots$. Then $f_0 = 1, f_2 = 2\frac{5}{6}, f_4 = 2\frac{139}{140} > f_2$.

Theorem 3. If sequence $\{\tilde{f}_k\}$ of figured approximants of BCF (1)–(2) whose elements satisfy conditions (12) and (16) converges, then sequece $\{f_k\}$ of ordinary approximants converges to the same limit.

Proof. Using the formulas (6)–(7), conditions (12) and (16) the following inequalities can be proved in much the same way as inequalities (19)–(20)

$$1 \leq Q_{2k,0}^{(p)} \leq 1 + a_{2k+1,1} + \frac{a_{2k+1,0}}{1 - |a_{2k+2,0}|}, \quad k = 1, 2, \dots, \quad p = 0, 1, \dots, \quad (27)$$

$$1 - |a_{2k,0}| \leq Q_{2k-1,0}^{(p)} \leq 1 + a_{2k,1}, \quad k = 1, 2, \dots, \quad p = 0, 1, \dots, \quad (28)$$

$$1 \leq Q_{0,2k}^{(p)} \leq 1 + a_{1,2k+1} + \frac{a_{0,2k+1}}{1 - |a_{0,2k+2}|}, \quad k = 1, 2, \dots, \quad p = 0, 1, \dots, \quad (29)$$

$$1 - |a_{0,2k}| \leq Q_{0,2k-1}^{(p)} \leq 1 + a_{1,2k}, \quad k = 1, 2, \dots, \quad p = 0, 1, \dots. \quad (30)$$

Using the formula (11), we consider the following differences $f_{4m+p} - \tilde{f}_{4m}, f_{4n+p-2} - \tilde{f}_{4n-2}, m, n, p = 1, 2, \dots$. We set $Z_{n,0}^{(1)} = 0, Z_{n,0}^{(2)} = 0,$

$$Z_{n,m}^{(1)} = \sum_{i=1}^m \frac{(-1)^i \left(F_{i,0}^{(n-i)} - F_{i,0}^{([\frac{m-i}{2}])} \right) \prod_{j=1}^i a_{j,0}}{\prod_{j=1}^i Q_{j,0}^{(n-j)} \tilde{Q}_{j,0}^{(m-j)}}, \quad m = 1, 2, \dots, \quad n > m,$$

$$Z_{n,m}^{(2)} = \sum_{i=1}^m \frac{(-1)^i \left(F_{0,i}^{(n-i)} - F_{0,i}^{([\frac{m-i}{2}])} \right) \prod_{j=1}^i a_{0,j}}{\prod_{j=1}^i Q_{0,j}^{(n-j)} \tilde{Q}_{0,j}^{(m-j)}}, \quad m = 1, 2, \dots, \quad n > m.$$

Then

$$f_{4m+p} - \tilde{f}_{4m} = F_{0,0}^{(4m+p)} - F_{0,0}^{(2m)} + Z_{4m+p,4m}^{(1)} + Z_{4m+p,4m}^{(2)} \\ + \frac{\prod_{j=1}^{4m+1} a_{j,0}}{\prod_{j=1}^{4m+1} \tilde{Q}_{j,0}^{(4m+p-j)} \prod_{j=1}^{4m} \tilde{Q}_{j,0}^{(4m-j)}} + \frac{\prod_{j=1}^{4m+1} a_{0,j}}{\prod_{j=1}^{4m+1} Q_{0,j}^{(4m+p-j)} \prod_{j=1}^{4m} \tilde{Q}_{0,j}^{(4m-j)}}.$$

From (23), (27)–(30) it follows that

$$Z_{4m+p,4m}^{(1)} = - \sum_{i=1}^m \frac{\left(F_{4i-3,0}^{(4m+p-4i+3)} - F_{4i-3,0}^{(2m-2i+1)} \right) \prod_{j=1}^{4i-3} |a_{j,0}|}{\prod_{j=1}^{4i-3} Q_{j,0}^{(4m+p-j)} \tilde{Q}_{j,0}^{(4m-j)}} \\ - \sum_{i=1}^m \frac{\left(F_{4i-2,0}^{(4m+p-4i+2)} - F_{4i-2,0}^{(2m-2i+1)} \right) \prod_{j=1}^{4i-2} |a_{j,0}|}{\prod_{j=1}^{4i-2} Q_{j,0}^{(4m+p-j)} \tilde{Q}_{j,0}^{(4m-j)}} + \sum_{i=1}^m \frac{\left(F_{4i-1,0}^{(4m+p-4i+1)} - F_{4i-1,0}^{(2m-2i)} \right) \prod_{j=1}^{4i-1} |a_{j,0}|}{\prod_{j=1}^{4i-1} Q_{j,0}^{(4m+p-j)} \tilde{Q}_{j,0}^{(4m-j)}} \\ + \sum_{i=1}^m \frac{\left(F_{4i,0}^{(4m+p-4i)} - F_{4i,0}^{(2m-2i)} \right) \prod_{j=1}^{4i} |a_{j,0}|}{\prod_{j=1}^{4i} Q_{j,0}^{(4m+p-j)} \tilde{Q}_{j,0}^{(4m-j)}} \\ = \sum_{i=1}^m \frac{\left| F_{4i-3,0}^{(4m+p-4i+3)} - F_{4i-3,0}^{(2m-2i+1)} \right| \prod_{j=1}^{4i-3} |a_{j,0}|}{\prod_{j=1}^{4i-3} Q_{j,0}^{(4m+p-j)} \tilde{Q}_{j,0}^{(4m-j)}} + \sum_{i=1}^m \frac{\left| F_{4i-2,0}^{(4m+p-4i+2)} - F_{4i-2,0}^{(2m-2i+1)} \right| \prod_{j=1}^{4i-2} |a_{j,0}|}{\prod_{j=1}^{4i-2} Q_{j,0}^{(4m+p-j)} \tilde{Q}_{j,0}^{(4m-j)}} \\ + \sum_{i=1}^m \frac{\left| F_{4i-1,0}^{(4m+p-4i+1)} - F_{4i-1,0}^{(2m-2i)} \right| \prod_{j=1}^{4i-1} |a_{j,0}|}{\prod_{j=1}^{4i-1} \tilde{Q}_{j,0}^{(4m+p-j)} \tilde{Q}_{j,0}^{(4m-j)}} + \sum_{i=1}^m \frac{\left| F_{4i,0}^{(4m+p-4i)} - F_{4i,0}^{(2m-2i)} \right| \prod_{j=1}^{4i} |a_{j,0}|}{\prod_{j=1}^{4i} Q_{j,0}^{(4m+p-j)} \tilde{Q}_{j,0}^{(4m-j)}} > 0.$$

Similarly $Z_{4m+p,4m}^{(2)} > 0$. Consequently

$$f_{4m+p} - \tilde{f}_{4m} > 0, \quad m, l = 0, 1, \dots, \quad p = 1, 2, \dots \quad (31)$$

Further,

$$f_{4n+p-2} - \tilde{f}_{4n-2} = F_{0,0}^{(4n+p-2)} - F_{0,0}^{(2n-1)} + Z_{4n+p-2,4n-2}^{(1)} + Z_{4n+p-2,4n-2}^{(2)} \\ + \frac{\prod_{j=1}^{4n-1} a_{j,0}}{\prod_{j=1}^{4n-1} Q_{j,0}^{(4n+p-2-j)} \prod_{j=1}^{4n-2} \tilde{Q}_{j,0}^{(4n-2-j)}} + \frac{\prod_{j=1}^{4n-1} a_{0,j}}{\prod_{j=1}^{4n-1} Q_{0,j}^{(4n+p-2-j)} \prod_{j=1}^{4n-2} \tilde{Q}_{0,j}^{(4n-2-j)}}.$$

From (25), (27)–(30) it follows that

$$\begin{aligned} Z_{4n+p-2,4n-2}^{(1)} = & - \sum_{i=1}^n \frac{\left(F_{4i-3,0}^{(4n+p-4i+1)} - F_{4i-3,0}^{(2n-2i)} \right) \prod_{j=1}^{4i-3} |a_{j,0}|}{\prod_{j=1}^{4i-3} Q_{j,0}^{(4n+p-2-j)} \tilde{Q}_{j,0}^{(4n-2-j)}} - \sum_{i=1}^n \frac{\left(F_{4i-2,0}^{(4n+p-4i)} - F_{4i-2,0}^{(2n-2i)} \right) \prod_{j=1}^{4i-2} |a_{j,0}|}{\prod_{j=1}^{4i-2} Q_{j,0}^{(4n+2p+l-2-j)} \tilde{Q}_{j,0}^{(4n-2-j)}} \\ & + \sum_{i=1}^{n-1} \frac{\left(F_{4i-1,0}^{(4n+p-4i-1)} - F_{4i-1,0}^{(2n-2i-1)} \right) \prod_{j=1}^{4i-1} |a_{j,0}|}{\prod_{j=1}^{4i-1} Q_{j,0}^{(4n+p-2-j)} \tilde{Q}_{j,0}^{(4n-2-j)}} + \sum_{i=1}^{n-1} \frac{\left(F_{4i,0}^{(4n+p-4i-2)} - F_{4i,0}^{(2n-2i-1)} \right) \prod_{j=1}^{4i} |a_{j,0}|}{\prod_{j=1}^{4i} Q_{j,0}^{(4n+p-2-j)} \tilde{Q}_{j,0}^{(4n-2-j)}} < 0. \end{aligned}$$

Similarly $Z_{4n+p-2,4n-2}^{(2)} < 0$. Consequently

$$f_{4n+p-2} - \tilde{f}_{4n-2} < 0, \quad n, p = 1, 2, \dots \quad (32)$$

Taking into account the inequalities (17), (31), (32) we obtain $\tilde{f}_{4m} < f_{4m+p+2} < \tilde{f}_{4m+2}$, $m = 0, 1, \dots$, $p = 1, 2, \dots$. In the case of convergence of sequence $\{\tilde{f}_{2k}\}$ we conclude that $\lim_{n \rightarrow \infty} \tilde{f}_n = \lim_{k \rightarrow \infty} f_k$. \square

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Антонова Т.М., Дмитришин М.В., Возна С.М. *Деякі властивості наближень гіллястих ланцюгових дробів спеціального вигляду з додатними і знакопозитивними частинними чисельниками* // Карпатські матем. публ. — 2018. — Т.10, №1. — С. 3–13.

Стаття присвячена дослідженню збіжності одного із узагальнень ланцюгових дробів — гіллястих ланцюгових дробів спеціального вигляду з двома гілками розгалужень. Такі дроби, так само як і двовимірні неперервні дроби та гіллясті ланцюгові дроби з двома нерівнозначними змінними, пов'язані з проблемою відповідності між формальним подвійним степеневим рядом і послідовністю раціональних наближень функції двох змінних.

На відміну від неперервних дробів, наближення яких будуються однозначно, існує багато способів побудови наближень гіллястих ланцюгових дробів загального та спеціального вигляду. У роботі розглянуто звичайні наближення та одну з конструкцій фігурних наближень досліджуваних гіллястих ланцюгових дробів, яка пов'язана із задачею відповідності.

Розглянуто деякі властивості наближень таких гіллястих ланцюгових дробів спеціального вигляду, частинні чисельники яких додатні і знакопозитивні, а частинні знаменники дорівнюють одиниці. Встановлено деякі необхідні і достатні умови фігурної збіжності. Доведено, що за сформульованих умов із збіжності послідовності фігурних наближень впливає збіжність послідовності звичайних наближень до тої самої границі.

Ключові слова і фрази: гіллястий ланцюговий дріб спеціального вигляду, звичайні наближення, фігурні наближення, збіжність, фігурна збіжність.



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THE NONLOCAL PROBLEM FOR THE $2n$ DIFFERENTIAL EQUATIONS WITH UNBOUNDED OPERATOR COEFFICIENTS AND THE INVOLUTION

We study a problem with periodic boundary conditions for a $2n$ -order differential equation whose coefficients are non-self-adjoint operators. It is established that the operator of the problem has two invariant subspaces generated by the involution operator and two subsystems of the system of eigenfunctions which are Riesz bases in each of the subspaces. For a differential-operator equation of even order, we study a problem with non-self-adjoint boundary conditions which are perturbations of periodic conditions. We study cases when the perturbed conditions are Birkhoff regular but not strongly Birkhoff regular or nonregular. We found the eigenvalues and elements of the system V of root functions of the operator which is complete and contains an infinite number of associated functions. Some sufficient conditions for which this system V is a Riesz basis are obtained. Some conditions for the existence and uniqueness of the solution of the problem with homogeneous boundary conditions are obtained.

Key words and phrases: operator of involution, differential-operator equation, eigenfunctions, Riesz basis.

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1 INTRODUCTION

The theory of differential equations with an unbounded operator coefficient was initiated by Hill and Yosida where the first theorems on the existence of the Cauchy problem solution for a linear homogeneous differential equation with respect to a function with values in a Banach space were obtained. Among works on this subject should be noted works of Kato T., Krein S.G., Mizohata S., Phillips R.S.

The boundary value problems for linear differential-operator equations are used in the simulation of boundary value problems for differential equations with partial derivatives, in particular, in the study of nonlocal problems. Significant results concerning the theory of boundary value problems for differential-operator equations were obtained in the papers of Vishik M.I., Boehner M., Gorbachuk V.I. and Gorbachuk M.L., Dezin O.O., Dubinsky Yu.V., Kochubei A.N., Lions J.-L., Mamedov K.S., Romanko V.K., Shakhmurov Veli B., Triebel Kh., Yakubov S., Yurchuk N.Yu.

During recent years the number of publications with the use of an involution operator in various sections of the theory of ordinary differential equations (see [2, 8–10, 12, 13, 15, 16]), partial differential equations (see [1, 7, 11, 14, 17, 18]) and differential equations with operator coefficients (see [3–6]) increased significantly.

In our article we will use the following notations. Let H be a separable Hilbert space and $A : D(A) \subset H \rightarrow H$ be the closed unbounded linear operator with the discrete spectrum

$\sigma(A) \equiv \{z_k \in \mathbb{R}, z_k = \alpha(k)^\gamma, \alpha, \gamma > 0, k = 1, 2, \dots\}$. We denote by $V(A) \equiv \{v_k \in H : k = 1, 2, \dots\}$ the system of the eigenfunctions of A which forms a Riesz basis in H , by $W(A) \equiv \{w_m \in H : m = 1, 2, \dots\}$ the biorthogonal system of the functions in the sense of equalities $(v_k, w_m; H) = 0, k \neq m, (v_k, w_k; H) = 1, k, m = 1, 2, \dots; H(A^s) \equiv \{h \in H : A^s h \in H\}, s \geq 0$. Let $H_1 \equiv L_2((0, 1), H)$ and $D_x : H_1 \rightarrow H_1$ is a strong derivative in the space H_1 ; $\left\| \frac{u(x+\Delta x) - u(x)}{\Delta x} - D_x u; H_1 \right\| \rightarrow 0, \Delta x \rightarrow 0$. Denote by $H_2 \equiv \{u \in H_1 : D_x^{2n} u \in H_1, A^{2n} u \in H_1\}$; by $[H]$ the algebra of the bounded linear operators $B : H \rightarrow H$. Denote by $H_0 \equiv L_2(0, 1)$; let I be the operator of the involution in the space $H_0, Iy(x) \equiv y(1-x)$, and let E be the identity transformation in $H_0, p_j \equiv \frac{1}{2}(E + (-1)^j I)$ are the orthoprojectors in the space $H_0, H_{0,j} \equiv \{y \in H_0 : y \equiv p_j y\}, j = 0, 1$. Let us denote by $W^{2n}(0, 1) \equiv \{y \in H_0 : y^{(m)} \in C[0, 1], m = \overline{0, 2n-1}, y^{(2n)} \in H_0\}$, by W^* the space of the continuous linear functionals on the space $W^{2n}(0, 1)$ and by $W_j^* \equiv \{l \in W^* : ly = 0, y \in H_{2,1-j} \cap W^{2n}(0, 1)\}; j = 0, 1$.

We consider the following boundary problem

$$Lw \equiv (-1)^n D_x^{2n} w(x) + A^{2n} w(x) + \sum_{s=1}^n a_s (D_x^{2s-1} w(x) + D_x^{2s-1} w(1-x)) = f(x), \quad x \in (0, 1), \quad (1)$$

$$\ell_j w \equiv D_x^{2j-1} w(0) - D_x^{2j-1} w(1) + l_j^1 w = \varphi_j, \quad (2)$$

$$\ell_{n+j} w \equiv D_x^{2j-2} w(0) - D_x^{2j-2} w(1) = \varphi_{n+j}, \quad (3)$$

where

$$\ell_j^1 w \equiv \sum_{r=0}^{m_j} (b_{j,r,0} D_x^r w(0) + b_{j,r,1} D_x^r w(1)), \quad j = 1, 2, \dots, n. \quad (4)$$

The function w is called the solution of the problem (1)–(4) if

$$\|Lw - f; H_1\| = 0, \quad \|l_j w - \varphi_j; H(A^{\beta_j})\| = 0,$$

$$\beta_{n+j} = 2n - 2j + \frac{3}{2}, \quad \beta_j = 2n - \max(m_j, 2j - 1) - \frac{1}{2},$$

$$a_j, b_{j,r,s} \in \mathbb{R}, \quad r = 0, 1, \dots, \quad m_j \leq 2n - 1, \quad s = 0, 1, \quad j = 1, 2, \dots, n.$$

The paper is arranged as follows. In Section 2 we investigate the properties of the operator of problem with periodic conditions for the equation $(-1)^n y^{(2n)} = \lambda y$. In Section 3 we study the spectral properties of the operator of a problem with boundary conditions that are periodic perturbations. In Sections 4 we construct a commutative group of operators that map the root functions of the operators of perturbed boundary-value problems. In Section 5 using these operators, systems of root functions of boundary-value problem operators are constructed and conditions for the completeness and basis property of these systems are established. In Section 6 some analogous results are obtained for the operators of boundary problems generated by differential equations with an involution.

2 A SPECTRAL PROBLEM WITH PERIODIC BOUNDARY CONDITIONS FOR A DIFFERENTIAL-OPERATOR EQUATION

Consider the partial case of the problem (1)–(4) with $a_j = 0, b_{j,r,s} = 0, r = 0, 1, \dots, m_j, s = 0, 1, j = 1, 2, \dots, n$, namely

$$(-1)^n D_x^{2n} u(x) + A^{2n} u(x) = f(x), \quad x \in (0, 1), \quad (5)$$

$$\ell_{0,j} u \equiv D_x^{2j-1} u(0) - D_x^{2j-1} u(1) = 0, \quad (6)$$

$$\ell_{0,n+j} u \equiv D_x^{2j-2} u(0) - D_x^{2j-2} u(1) = 0, \quad j = 1, 2, \dots, n. \quad (7)$$

Let L_0 be the operator of the problem (5)–(7),

$$L_0 u \equiv (-1)^n D_x^{2n} u + A^{2n} u, \quad u \in D(L_0), \quad D(L_0) \equiv \{u \in H_2 : l_{0,j} u = 0, j = 1, 2, \dots, 2n\}.$$

Consider the spectral problem

$$(-1)^n D_x^{2n} u(x) + A^{2n} u(x) = \lambda u(x), \quad \lambda \in \mathbb{C}, \quad (8)$$

$$l_{0,j} u \equiv u^{(2j-1)}(0) - u^{(2j-1)}(1) = 0, \quad j = 1, 2, \dots, 2n. \quad (9)$$

We find the solution of the spectral problem (8), (9) as the product $u(x) = y(x)v_k$, $v_k \in V(A)$, $k = 1, 2, \dots$.

To determine the unknown function $y \in W^{2n}(0, 1)$ we obtain the spectral problem

$$(-1)^n y^{(2n)}(x) + z_k^{2n} y(x) = \lambda y(x), \quad \lambda \in \mathbb{C}, \quad (10)$$

$$l_{0,j} y \equiv y^{(2j-1)}(0) - y^{(2j-1)}(1) = 0, \quad j = 1, 2, \dots, n, \quad (11)$$

$$l_{0,n+j} y \equiv y^{(2j-2)}(0) - y^{(2j-2)}(1) = 0, \quad j = 1, 2, \dots, n. \quad (12)$$

Let $L_{0,k}$ be the operator of the problem (10)–(12),

$$L_{0,k} y \equiv (-1)^n y^{(2n)} + z_k^{2n} y, \quad y \in D(L_{0,k}), \quad D(L_{0,k}) \equiv \{y \in W^{2n}(0, 1) : l_{0,j} y = 0, j = \overline{1, 2n}\}.$$

The roots ρ_j of the characteristic equation $(-1)^n \rho^{2n} = \lambda - z_k^{2n}$ of the differential equation

$$(-1)^n y^{(2n)}(x) + z_k^{2n} y(x) = \lambda y(x), \quad (13)$$

are defined by the relations

$$\rho_j = \omega_j \rho, \quad \omega_1 = i, \quad \omega_j = i \exp\left(\frac{i\pi(j-1)}{2n}\right), \quad j = 2, 3, \dots, n.$$

The fundamental system of the solutions of the differential equation (13) is defined by the formulas

$$Y_j(x, \rho) \equiv \frac{1}{2}(\exp \omega_j \rho x + \exp \omega_j \rho(1-x)) \in H_{2,0}, \quad j = 1, 2, \dots, n, \quad (14)$$

$$Y_{n+j}(x, \rho) \equiv \frac{1}{2}(\exp \omega_j \rho x - \exp \omega_j \rho(1-x)) \in H_{2,1}, \quad j = 1, 2, \dots, n. \quad (15)$$

Substituting the general solution

$$y(x, \rho) = \sum_{s=1}^{2n} C_s Y_s(x, \rho)$$

of the differential equation (13) into the boundary conditions (11), (12) we obtain the equation for determination the eigenvalues of the operator $L_{0,k}$

$$\Delta(\rho) = \det(l_r Y_j)_{r,j=1}^{2n} = 0. \quad (16)$$

By substituting the functions (14), (15) in the boundary conditions (11), (12), we obtain $l_{0,r} Y_{n+j} = 0$, $l_{0,n+r} Y_j = 0$, $j, r = 1, 2, \dots, n$. Therefore $\Delta(\rho) = \Delta_0(\rho)\Delta_1(\rho) = 0$, where $\Delta_s(\rho) = \det(l_{sn+r} Y_{sn+j})_{r,j=1}^n$, $s = 0, 1$.

The operator $L_{0,k}$ is self-adjoint (see [15]). Therefore the solutions of the equation (16) are $\rho_q = 2q\pi i$, $q = 0, 1, 2, \dots$, which are numbered in ascending order and lie on the half-line $Im\rho = 0$, $Re\rho \geq 0$.

Thus, the operator $L_{0,k}$ has eigenvalues $\lambda_{q,k} = (\rho_q)^{2n} + z_k^{2n}$, $q = 0, 1, \dots$. We obtain the following result.

Lemma 2.1. *The self-adjoint operator $L_{0,k}$ has a point spectrum*

$$\sigma(L_{0,k}) = \left\{ \lambda_q \in \mathbb{R} : \lambda_q = (2\pi q)^{2n} + z_k^{2n}, q = 0, 1, \dots \right\}$$

and a system of eigenfunctions

$$V(L_{0,k}) \equiv \left\{ v_q(x) \in L_2(0,1) : v_0(x) = 1, v_{2q}(x) \equiv \sqrt{2} \cos 2\pi q x, \right. \\ \left. v_{2q-1}(x) \equiv \sqrt{2} \sin 2\pi q x, q = 1, 2, \dots \right\},$$

which is an orthonormal basis of the space H_0 .

Remark 2.1. *The systems*

$$V_0(L_{0,k}) \equiv \{v_{2q}(x) : q = 0, 1, \dots\}, V_1(L_{0,k}) \equiv \{v_{2q-1}(x) : q = 1, 2, \dots\}$$

form an orthonormal basis in spaces $H_{0,0}$ and $H_{0,1}$, respectively.

Therefore, the operator L_0 has the following eigenfunctions in the space H_1

$$V(L_0) \equiv \left\{ v_{q,k}(x, L_0) \in H_1 : v_{q,k}(x, L_0) \equiv v_q(x)v_k, q = \overline{0, \infty}, k = \overline{1, \infty} \right\}.$$

A system of functions $\{h_s\}_{s=1}^{\infty} \subset H$ is called a Riesz basis in a Hilbert space H , if $\{h_s\}_{s=1}^{\infty}$ is complete in the space H , and for any orthonormal basis $\{e_s\}_{s=1}^{\infty} \subset H$ there exists an isomorphism $B : H \rightarrow H$, $Be_s = h_s$, $s = 1, 2, \dots$.

The product of a system $V(A)$ and an orthonormal system $V(L_{0,k})$ is the Riesz basis (see [9]) in the space H_1 . Thus, the following theorem is true.

Theorem 1. *The operator L_0 has a discrete spectrum*

$$\sigma(L_0) \equiv \left\{ \lambda_{q,k} \in \mathbb{R} : \lambda_{q,k} \equiv \rho_q^{2n} + z_k^{2n}, k = \overline{1, \infty}, q = \overline{0, \infty} \right\},$$

and the system of the eigenfunctions $V(L_0)$ forms the Riesz basis in the space H_1 .

Let us consider the functions

$$y_r(x, \rho_q) \equiv \frac{1}{2} (1 + e^{\omega_r \rho_q})^{-1} \left(e^{\omega_r \rho_q x} + e^{\omega_r \rho_q (1-x)} \right), \\ y_1(x, \rho_q) \equiv \frac{1}{2} (1 - 2x) \sin \rho_q x, \quad r = 2, 3, \dots, n, q = 1, 2, \dots, \quad (17)$$

and determine the square matrix

$$B_{0,p}(x, \rho_q) \equiv (\beta_{p,s}^0)_{p,s=1}^n$$

of the order n according to the following: the row with number p is determined by the elements of the system (17) $\beta_{p,s}^0(x, \rho_q) \equiv y_s(x, \rho_q)$ and the other lines by the formulas $\beta_{j,s}^0(x, \rho_q) \equiv (\omega_s)^{2j-1}$ with $j \neq p$, $j, s = 1, 2, \dots, n$.

We denote the determinant of the matrix $B_{0,p}(x, \rho_q)$ by $y_{1,p}(x, \rho_q)$.

Substituting the determinant into conditions (11), (12), we obtain

$$l_{0,r} y_{1,p} = 0, \quad r \neq p, \quad l_{0,p} y_{1,p} = (\rho_q)^{2p-1} h(i) W^n, \quad (18)$$

where the Vandermonde determinant W^n is constructed by the numbers

$$1, (\omega_2)^2, \dots, (\omega_n)^2, h(i) = (-i)^{n-1} i, i = \sqrt{-1}, r = 1, \dots, 2n, m = 1, 2, \dots, n.$$

Consider the functions $y_{2,p}(x, \rho_q) \equiv (h(i)W^n)^{-1} y_{1,p}(x, \rho_q)$, $p = 1, 2, \dots, n$.

From the relation (18) we obtain

$$l_{0,r}y_{2,p} = 0, r \neq p, l_{0,p}y_{2,p} = (\rho_q)^{2p-1}, p, r = 1, 2, \dots, n. \quad (19)$$

Similarly, let us consider the system of functions

$$\begin{aligned} y_{n+r}(x, \rho_q) &\equiv \frac{1}{2} (1 - e^{w_r \rho_q})^{-1} (e^{w_r \rho_q x} - e^{w_r \rho_q (1-x)}), \\ y_{n+1}(x, \rho_q) &\equiv \frac{1}{2} (1 - 2x) \cos \rho_q x, \quad r = 2, 3, \dots, n, q = 1, 2, \dots, \end{aligned} \quad (20)$$

and a square matrix

$$B_{1,r}(x, \rho_q) \equiv (\beta_{p,s}^1)_{p,s=1}^n$$

of the order n which rows are determine by following: the row with number r is determined by the elements of the system (20) $\beta_{r,s}^1(x, \rho_q) \equiv y_{n+s}(x, \rho_q)$ and the other lines by the equalities

$$\beta_{j,s}^1(x, \rho_q) \equiv (\omega_s)^{2j-2}, j \neq r, r, s = 1, 3, \dots, n.$$

We denote the determinant of the matrix by $y_{1,n+r}(x, \rho_q)$.

Substituting it into conditions (11), (12), we get

$$l_{0,j}y_{1,n+r} = 0, j \neq n+r, l_{0,n+r}y_{1,n+r} = W^n (\rho_q)^{2r-2}. \quad (21)$$

Let us define the functions $y_{2,n+r}(x, \rho_q) \equiv (W^n)^{-1} y_{1,n+r}(x, \rho_q)$, $r = 1, 2, \dots, n$.

Taking the relation (21) for the functions $y_{2,n+r}(x, \rho_q)$ into account, we obtain

$$l_{0,j}y_{2,n+r} = 0, j \neq n+r, l_{0,n+r}y_{2,n+r} = (\rho_q)^{2r-2}, j = 1, \dots, 2n, r = 1, 2, \dots, n.$$

Remark 2.2. *There exist positive numbers K_0, K_1 such that*

$$K_1 \leq \|y_{2,j}(x, \rho_q); H_0\| \leq K_2 < \infty, \quad j = 1, 2, \dots, 2n, q = 1, 2, \dots. \quad (22)$$

Here $K_s, s \in \mathbb{N}$, are positive constants.

3 NONLOCAL BOUNDARY VALUE PROBLEM

For the differential-operator equation (5) and an arbitrary fixed $p \in \{1, 2, \dots, n\}$ and $b \in \mathbb{R}$ we consider the boundary value problem

$$\ell_{1,j}u \equiv D_x^{2j-1}u(0) - D_x^{2j-1}u(1) = 0, \quad j \neq p, j = 1, 2, \dots, n, \quad (23)$$

$$\ell_{1,p}u \equiv D_x^{2p-1}u(0) - D_x^{2p-1}u(1) + l_p^2 u = 0, \quad (24)$$

$$\ell_{1,n+j}u \equiv D_x^{2j-2}u(0) - D_x^{2j-2}u(1) = 0, \quad j = 1, 2, \dots, n, \quad (25)$$

with

$$\ell_p^2 u \equiv b(D_x^{2p-1}u(0) + D_x^{2p-1}u(1)) = 0, \quad b \in \mathbb{R}. \quad (26)$$

We will use following notations. Let L_1 be the operator of the problem (5), (23)–(26) and $L_1 u \equiv (-1)^n D_x^{2n} u(x) + A^{2n} u(x)$, $u \in D(L_1)$, $D(L_1) \equiv \{u \in H_2 : l_{1,r} u = 0, r = \overline{1, 2n}\}$.

We find the solution of the spectral problem (8), (23)–(26) as the product $u(x) = y(x)v_k$, $v_k \in V(A)$, $k = 1, 2, \dots$.

To determine the unknown function $y \in W^{2n}(0, 1)$ we consider the spectral problem

$$(-1)^n y^{(2n)}(x) + z_k^{2n} y(x) = \lambda y(x), \quad \lambda \in \mathbb{C}, \quad (27)$$

$$\ell_{1,j} y \equiv y^{(2j-1)}(0) - y^{(2j-1)}(1) = 0, \quad j \neq p, \quad j = 1, 2, \dots, n, \quad (28)$$

$$\ell_{1,p} y \equiv y^{(2p-1)}(0) - y^{(2p-1)}(1) + l_p^2 y = 0, \quad (29)$$

$$l_{1,n+j} y \equiv y^{(2j-2)}(0) - y^{(2j-2)}(1) = 0, \quad j = 1, 2, \dots, n, \quad (30)$$

with

$$l_p^2 y \equiv b \left(y^{(2p-1)}(0) + y^{(2p-1)}(1) \right). \quad (31)$$

Let $L_{1,k} \equiv L_{1,k,p}$ be the operator of the problem (27)–(31) and

$$L_{1,k} y \equiv (-1)^n y^{(2n)}(x) + z_k^{2n} y(x), \quad y \in D(L_{1,k}), \quad D(L_{1,k}) \equiv \{y \in W^{2n}(0, 1) : l_{1,j} y = 0, j = \overline{1, 2n}\}.$$

Let $V(L_{1,k})$ be the system of root functions of the operator $L_{1,k}$, let $R(L_{1,k}) \equiv E + S(L_{1,k})$ be the operator which maps the system $V(L_{0,k})$ into the system $V(L_{1,k})$.

Theorem 2. For any $b \in \mathbb{R}$, $p \in \{1, 2, \dots, n\}$, the operator $L_{1,k}$ has the point spectrum $\sigma(L_{0,k})$ and the system of root functions $V(L_{1,k})$ forms a Riesz basis in H_0 .

Proof. We will show that eigenvalues of the operators $L_{0,k}$ and $L_{1,k}$ coincide.

We substitute the fundamental system (14), (15) for the solutions of the differential equation (27) into the boundary conditions (28)–(31). Using $l_p^2 y_{n+j}(x, \rho) = 0$, $j, p = \overline{1, n}$, we obtain the same equations for determination the spectrum

$$\det(l_{1,j} y_r(x, \rho))_{j,r=1}^{2n} = \det(l_{1,j} y_r(x, \rho))_{j,r=1}^n \det(l_{1,n+j} y_{n+r}(x, \rho))_{j,r=1}^n.$$

Let us define elements of the system $V(L_{1,k})$.

It is easy to see that $v_{2q}(x) \in D(L_{1,k})$, $L_{1,k} v_{2q}(x) = \lambda_q v_{2q}(x)$, $q = 0, 1, 2, \dots$. Hence

$$v_{2q}(x, L_{1,k}) \equiv v_{2q}(x), \quad q = 0, 1, \dots \quad (32)$$

We define the root functions of the operator $L_{1,k}$ as

$$v_{2q-1}(x, L_{1,k}) \equiv v_{2q-1}(x) + c_{b,p} y_{2,p}(x, \rho_q). \quad (33)$$

Substituting (33) into the boundary condition (29) and taking the equality (19) into account, we obtain $c_{b,p} = -\sqrt{2}b$,

$$v_{2q-1}(x, L_{1,k}) \equiv v_{2q-1}(x) - \sqrt{2}b y_{2,p}(x, \rho_q), \quad q = 1, 2, \dots \quad (34)$$

Thus, the operator $L_{1,k}$ has a system of root functions (32)–(34) in the sense of equalities

$$\begin{aligned} L_{1,k} v_{2q-1}(x, L_{1,k}) &= \lambda_{q,k} v_{2q-1}(x, L_{1,k}) + \xi_{b,q} v_{2q}(x, L_{1,k}), \\ \xi_{b,q} &= -4\sqrt{2}bn(\rho_q)^{2n-1}, \quad L_{1,k} v_{2q}(x, L_{1,k}) = \lambda_{q,k} v_{2q}(x, L_{1,k}), \quad q = 1, 2, \dots \end{aligned}$$

Given the regularity according to Birkhoff (see [15]) of boundary conditions (28)–(31) we obtain that the system $V(L_{1,k})$ is complete and minimal in the space H_0 . \square

Let W^{n-1} be the Vandermonde determinant constructed by elements $\omega_2^2, \omega_3^2, \dots, \omega_n^2$; let $R_1 \equiv E + S_1$ be the operator which maps the system $V(L_0)$ into the system V_1 and elements of this system are

$$\begin{aligned} v_{2q-1,1}(x) &\equiv (1 - \sqrt{2}^{-1} W^n (h(i) W^{n-1})^{-1} b (1 - 2x)) v_{2q-1}(x), \\ v_{0,1}(x) &\equiv v_0(x), \quad v_{2q,1}(x) \equiv v_{2q}(x), \quad q = 1, 2, \dots \end{aligned} \quad (35)$$

Lemma 3.1. *The system V_1 forms a Riesz basis in H_0 .*

Proof. For an arbitrary function $\varphi \in H_0$ we have

$$\begin{aligned} \varphi &= \varphi_0 v_0(x) + \sum_{q=0}^{\infty} (\varphi_{2q} v_{2q}(x) + \varphi_{2q-1} v_{2q-1}(x)) \in H_0, \\ \|\varphi; H_0\|^2 &= |\varphi_0|^2 + \sum_{q=1}^{\infty} (|\varphi_{2q}|^2 + |\varphi_{2q-1}|^2) < \infty, \end{aligned}$$

consider the function

$$\begin{aligned} \varphi_1 &= R_1 \varphi = \varphi_0 v_{0,1}(x) + \sum_{q=1}^{\infty} (\varphi_{2q} v_{2q,1}(x) + \varphi_{2q-1} v_{2q-1,1}(x)), \\ \|R_1 \varphi; H_0\|^2 &\leq K_3 \|\varphi; H_0\|^2, \quad K_3 = 2 \left(1 + |W^n (W^{n-1})^{-1} b|^2 \right). \end{aligned}$$

Therefore $\|R_1; [H_0]\|^2 \leq K_3 < \infty$, $R_1 \equiv E + S_1 \in [H_0]$, $R_1^{-1} \equiv E - S_1 \in [H_0]$. Taking into account the Bari Theorem (see [9]), we obtain the following statement: the system V_1 forms the Riesz basis in H_0 . \square

Therefore, the operator L_1 has the following system of root functions in the space H_1

$$V(L_1) \equiv \left\{ v_{q,k}(x, L_1) \equiv v_q(x, L_{1,k}) \quad v_k \in H_1 : q = \overline{0, \infty}, k = \overline{1, \infty} \right\}.$$

Remark 3.1. *The operator L_1 has a system of root functions in the means of equalities*

$$\begin{aligned} L_1 v_{2q-1,k}(x, L_1) &= \lambda_{q,k} v_{2q-1,k}(x, L_1) + \xi_{b,q} v_{2q,k}(x, L_1), \\ \xi_{b,q} &= -4\sqrt{2}bn (\rho_q)^{2n-1}, \quad q, k = 1, 2, \dots, \\ L_1 v_{2q,k}(x, L_1) &= \lambda_{q,k} v_{2q,k}(x, L_1), \quad q = 0, 1, \dots, k = 1, 2, \dots \end{aligned}$$

Theorem 3. *For any fixed numbers $p \in \{1, 2, \dots, n\}$, $b \in \mathbb{R}$, the system $V(L_1)$ is the Riesz basis of the space H_1 .*

Proof. Let $R(L_{1,k}) \equiv E + S(L_{1,k}) : V(L_{0,k}) \rightarrow V(L_{1,k})$, let p_k be a projection in H , $p_k y \equiv (y, w_k(A); H) v_k$, $R(L_1) \equiv \sum_{k=1}^{\infty} R(L_{1,k}) p_k$.

From the definition of the operator $R(L_1) = E + S(L_1)$ it follows that $R^{-1}(L_1) = E - S(L_1)$.

Therefore the system $V(L_1)$ is complete and minimal in the space H_1 . Taking into account the representations of the elements of the system $V(L_{1,k})$ and Theorem 2, we obtain $\|R(L_1); [H_1]\| \leq K_4 \|R(L_{1,k}); [H_0]\| < \infty$.

Taking into account the Bari Theorem (see [9]), we obtain the following statement: the system V_1 forms the Riesz basis in H_1 . \square

4 TRANSFORMATION OPERATORS

For any $k \in \mathbb{N}$, $p \in \{1, 2, \dots, n\}$, we define the operator $B_p : H_0 \rightarrow H_0$ as the operator whose eigenvalues coincide with eigenvalues of the operator $L_{0,k}$, and the root functions are defined by

$$v_{2s}(x, B_p) \equiv v_{2s}(x), \quad v_{2q-1}(x, B_p) \equiv v_{2q-1}(x) + c_q(B_p)y_{2,p}(x, \rho_q), \quad (36)$$

where $c_q(B_p) \in \mathbb{R}$, $s = 0, 1, \dots$, $q = 1, 2, \dots$.

The operator which maps the system $V(L_{0,k})$ into the system $V(B_p)$ of the root functions of the operator B_p is denoted by $R(B_p) \equiv E + S(B_p)$, where $S(B_p) : H_{0,0} \rightarrow H_{0,1}$, $S(B_p) : H_{0,1} \rightarrow 0$.

We denote by $G_{0,p}(L_{0,k}) \equiv \{R(B_p)\}$ such that the root functions of the operator B_p are defined by the equalities (36), and $G_{0,p,c}(L_{0,k}) \equiv G_{0,p}(L_{0,k}) \cap [H_0]$.

Remark 4.1. Using that $S(B_p) : H_{0,0} \rightarrow H_{0,1}$, $S(B_p) : H_{0,1} \rightarrow 0$ we obtain $S^2(B_p) = 0$, $R^{-1}(B_p) \equiv E - S(B_p)$.

Consequently, the operator $R(B_p)$ has a dense domain in the space H_0 and the system of root functions is complete and minimal in H_0 .

Similarly, using the root functions of an adjoint operator $L_{1,k}^*$, we define the functions

$$w_0(x, B_p) \equiv v_0(x) + c_0(1 - 2x), \\ w_{2q}(x, B_p) \equiv v_{2q}(x) + c_q(B_p)y_{2,2n-p-1}(x, \rho_q), \quad w_{2q-1}(x, B_p) \equiv v_{2q-1}(x), \quad q = 1, 2, \dots,$$

and the set of operators $G_{1,p}(L_{0,k}) \equiv \left\{ R(B_p^*) = E + S(B_p^*), R(B_p) \in G_{0,p}(L_{0,k}) \right\}$.

Theorem 4. For any $b \in \mathbb{R}$, $p \in \{1, 2, \dots, n\}$, the operator B_p has the point spectrum $\sigma(L_{0,k})$ and the system of root functions $V(L_{1,k})$ forms the Riesz basis in H_0 if and only if the sequence $c_q(B_p)$ is bounded, i.e. $|c_q(B_p)| \leq K_5 < \infty$, $q = 1, 2, \dots$.

Proof. The necessity. Let the system $V(B_p)$ be the Riesz basis in H_0 , i.e. $R(B_p) \in [H_0]$, then $S(B_p) = E - R_p(B) \in [H_0]$. From the definition of the operator B_p we have

$$S(B_p)v_{2q-1}(x) = c_q(B_p)y_{2,p}(x, \rho_q), \quad q = 1, 2, \dots$$

Therefore, taking the estimate (22) into account, we obtain

$$|c_q(B_p)| \leq \|S(B_p); [H_0]\| \|y_{2,p}(x, \rho_q); H_0\|^{-1} \leq K_6 < \infty, \\ K_6 = K_5^{-1} \|S(B_p); [H_0]\|, \quad q = 1, 2, \dots$$

The sufficiency. The completeness of the system $V(B_p)$ in the space H_0 follows from Remark 4.1.

Let $\varphi \in H_0$, $\varphi = \varphi_0 + \varphi_1$, $\varphi_s \in H_{0,s}$, $s = 0, 1$. Then we have

$$\varphi = \varphi_0 v_0(x) + \sum_{q=1}^{\infty} (\varphi_{2q} v_{2q}(x) + \varphi_{2q-1} v_{2q-1}(x)) \in H_0,$$

$$\|\varphi; H_0\|^2 = |\varphi_0|^2 + \sum_{q=1}^{\infty} (|\varphi_{2q}|^2 + |\varphi_{2q-1}|^2) < \infty,$$

$$R(B_p)\varphi = \varphi_0 v_0(x, B_p) + \sum_{q=1}^{\infty} (\varphi_{2q} v_{2q}(x, B_p) + \varphi_{2q-1} v_{2q-1}(x, B_p)) \in H_0,$$

$$R(B_p)\varphi = \varphi_0 v_0(x, B_p) + \sum_{q=1}^{\infty} (\varphi_{2q} v_{2q}(x) + \varphi_{2q-1} c_q(B_p) (v_{2q-1}(x, L_{1,k}) - v_{2q-1}(x))),$$

$$\|R(B_p)\varphi; H_0\|^2 \leq K_7 \|\varphi; H_0\|^2, \quad K_7 = 3 \left(1 + K_6^2 + K_6^2 \|R(L_{1,k}); [H_0]\|^2 \right).$$

Therefore, $\|R(B_p); [H_0]\|^2 \leq K_7 < \infty$.

Consider equalities $R(B_p) = E + S(B_p)$, $R^{-1}(B_p) = E - S(B_p)$. We have

$$R^{-1}(B_p) = 2E - R(B_p).$$

Therefore, $\|R^{-1}(B_p); [H_0]\|^2 \leq K_8 < \infty$, $K_8 = 8 + 2K_7$. Taking into account the Bari Theorem (see [9]), we obtain that the system V_1 forms the Riesz basis in H_0 . \square

Suppose that $Q_0(I)$ is a set of operators $R = E + S$, such that $S : H_{0,0} \rightarrow H_{0,1}$, $S : H_{0,1} \rightarrow 0$, $Q_{0,c}(I) \equiv [H_0] \cap Q_0(I)$. Using that $S^2(B_p) = 0$, $R(B_p) \in G_{0,p}(L_{0,k}) \subset Q_0(I)$ on the set $Q_0(I)$, we can define the operation of multiplication

$$R_1 R_2 \equiv (E + S_1)(E + S_2) = E + S_1 + S_2, \quad R_1, R_2 \in Q_0(I).$$

In particular, $(E + S)(E - S) = E - S^2 = E$, $R = E + S \in Q_0(I)$.

Therefore, for each operator $R = E + S \in Q_0(I)$ there exists a unique inverse operator $R^{-1} = E - S$.

According to the definition of the operator B_p and the set $G_{0,p}(L_{0,k})$ we have the inclusions

$$G_{0,p}(L_{0,k}) \subset Q_0(I), \quad G_{c,0,p}(L_{0,k}) \subset Q_{0,c}(I), \quad p \in \{1, 2, \dots, n\}.$$

Thus, the set $Q_0(I)$ is an Abelian group which contains the Abelian subgroups $Q_{c,0}(I)$, $G_{0,p}(L_{0,k})$, $G_{c,p}(L_{0,k})$, $p \in \{1, 2, \dots, n\}$. Therefore, for all operators $R_j = E + S_j \in Q_0(I)$, $j = 1, 2, \dots, d$, $d \in \mathbb{N}$, the following equality

$$\prod_{j=1}^d R_j \equiv \prod_{j=1}^d (E + S_j) = E + \sum_{j=1}^d S_j, \quad d \in \mathbb{N}, \quad (37)$$

holds.

5 NONLOCAL BOUNDARY VALUE PROBLEMS FOR A DIFFERENTIAL-OPERATOR EQUATION

5.1. For the differential-operator equation (5) and arbitrary fixed indices $b_{p,r,s} \in \mathbb{R}$, $p \in \{1, 2, \dots, n\}$, $r = 0, 1, \dots, k_j$, $s = 0, 1$, $j = 1, 2, \dots, n$, we consider the boundary problem generated by nonlocal conditions

$$\ell_{2,j} w \equiv D_x^{2j-1} w(0) - D_x^{2j-1} w(1) = 0, \quad j \neq p, \quad (38)$$

$$\ell_{2,p} w \equiv D_x^{2p-1} w(0) - D_x^{2p-1} w(1) + l_p^1 w = 0, \quad (39)$$

$$\ell_{2,n+j} w \equiv D_x^{2j-2} w(0) - D_x^{2j-2} w(1) = 0, \quad j = 1, 2, \dots, n, \quad (40)$$

where

$$\ell_p^1 w \equiv \sum_{r=0}^{m_p} (b_{p,r,0} D_x^r w(0) + b_{j,r,1} D_x^r w(1)). \quad (41)$$

Assumption P₁: $b_{p,r,0} = (-1)^{r+1} b_{p,r,1}$, $r = 0, 1, \dots, m_p$, $j, p = 1, 2, \dots, n$.

Assumption P₂: $m_p \leq 2p - 1$, $p = 1, 2, \dots, n$.

Remark 5.1. *Assumption P₁ implies that $l_p^1 \in W_1^*$, $p = 1, 2, \dots, n$.*

Let $L_2 \equiv L_{2,p}$ be the operator of the problem (5), (38)–(41) and

$$L_2 u \equiv (-1)^n D_x^{2n} u(x) + A^{2n} u(x), \quad u \in D(L_2), \quad D(L_2) \equiv \{u \in H_2 : l_{2,j} u = 0, j = 1, 2, \dots, 2n\}.$$

The solution of the spectral problem (5), (38)–(41) is defined as the product $w(x) = y(x)v_k$, $v_k \in V(A)$, $k = 1, 2, \dots$.

To determine the unknown function $y \in W^{2n}(0, 1)$ we consider the spectral problem

$$(-1)^n y^{(2n)}(x) + z_k^{2n} y(x) = \lambda y(x), \quad \lambda \in \mathbb{C}, \quad (42)$$

$$l_{2,j} y \equiv y^{(2j-1)}(0) - y^{(2j-1)}(1) = 0, \quad j \neq p, \quad (43)$$

$$l_{2,p} y \equiv y^{(2p-1)}(0) - y^{(2p-1)}(1) + l_p^1 y = 0, \quad (44)$$

$$l_{2,n+j} y \equiv y^{(2j-2)}(0) - y^{(2j-2)}(1) = 0, \quad j = 1, 2, \dots, n. \quad (45)$$

Let $L_{2,k} \equiv L_{2,k,p}$ be the operator of the problem (42)–(45) and

$$L_{2,k} y \equiv (-1)^n y^{(2n)}(x) + z_k^{2n} y(x),$$

$$y \in D(L_{2,k}), \quad D(L_{2,k}) \equiv \{y \in W^{2n}(0, 1) : l_{2,j} y = 0, j = 1, 2, \dots, 2n\}.$$

Theorem 5. Suppose that the Assumption P_1 holds. Then for arbitrary numbers $b_{p,r,s} \in \mathbb{R}$, $s = 0, 1$, $r = 0, 1, \dots, m_p$, $p \in \{1, 2, \dots, n\}$, the following statements hold

- 1) the eigenvalues of the operators $L_{0,k}$ and $L_{2,k}$ coincide;
- 2) the system $V(L_{2,k})$ is complete and minimal in the space H_0 ;
- 3) if in addition the Assumption P_2 holds, then the system $V(L_{2,k})$ is the Riesz basis of the space H_0 .

Proof. The proof of part 1 of the theorem can be made in the same way as in Theorem 2.

Let us define the elements of the system $V(L_{2,k})$. A direct substitution gives that the function $v_{2q}(x)$, $q = 0, 1, \dots$, satisfies the conditions (43)–(45). Therefore, the root function of the operator $L_{2,k}$ with respect to the eigenvalue $\lambda_{q,k}$ is defined by

$$\begin{aligned} v_{2q}(x, L_{2,k}) &= v_{2q}(x, L_{0,k}), \quad q = 0, 1, \dots, \\ v_{2q-1}(x, L_{2,k}) &= v_{2q-1}(x) + c_{q,p} y_{2,p}(x, \rho_q), \\ c_{q,p} &= -l_p^1 (v_{2q-1}(x)) (l_{2,p} y_{2,p}(x, \rho_q))^{-1}, \quad q = 1, 2, \dots \end{aligned}$$

Consequently $L_{2,k} \in Q_0(I)$. If the Assumption P_2 holds, then from the inequality $|l_{p,b}^1 v_{2q-1}| \leq K_9 (\rho_q)^{2p-2}$ we obtain the inequality

$$|l_p^1 (v_{2q-1}(x)) (l_{2,n+p} y_{2,p}(x, \rho_{0,q}))^{-1}| \leq K_{10} < \infty. \quad (46)$$

Taking Theorem 4 into account, we obtain the third statement of the theorem. \square

Therefore, the operator L_2 has the following system of root functions in the space H_1

$$V(L_2) \equiv \left\{ v_{q,k}(x, L_2) \equiv v_q(x, L_{2,k}) v_k : q = \overline{0, \infty}, k = \overline{1, \infty} \right\}.$$

Remark 5.2. The operator L_2 has a system of root functions in the means of equalities

$$L_2 v_{2q-1,k}(x, L_2) = \lambda_{q,k} v_{2q-1,k}(x, L_2) + \xi_{q,p} v_{2q,k}(x, L_2), \quad (47)$$

$$\xi_{q,p} = -4\sqrt{2} (\rho_q)^{2n-1} c_{q,p}, \quad q, k = 1, 2, \dots, \quad (48)$$

$$L_2 v_{2q,k}(x, L_2) = \lambda_{q,k} v_{2q,k}(x, L_2), \quad q = 0, 1, \dots, k = 1, 2, \dots \quad (49)$$

Theorem 6. *Suppose that the Assumption P_1 holds. Then, for arbitrary numbers $b_{p,r,0} \in \mathbb{R}$, $r = 0, 1, \dots, m_p$, $p \in \{1, 2, \dots, n\}$, the following statements hold*

- 1) *the eigenvalues of the operators L_0 and L_2 coincide;*
- 2) *the system $V(L_2)$ is complete and minimal in the space H_1 ;*
- 3) *if in addition the Assumption P_2 holds, then the system $V(L_2)$ forms the Riesz basis of the space H_1 .*

Proof. Taking Theorem 5 into account, it is possible to determine the elements of a system $W(L_{2,k})$ which is biorthogonal to the system $V(L_{2,k})$ in the space H_0 .

Therefore, there exists $W(L_2) \equiv \{w_{q,k}(x, L_2)w_k : q = 0, 1, \dots, k = 1, 2, \dots\}$ which is the biorthogonal system of functions to the system $V(L_2)$ in the space H_1 .

Thus the second statement of the theorem is proved.

Suppose that the Assumption P_2 holds. Taking the inequalities (46) into account, we obtain the estimate

$$\|R(L_2); [H_1]\| \leq K_{11} < \infty.$$

From the Bari Theorem (see [9]) we obtain the statement: the system V_1 forms the Riesz basis in H_1 . \square

5.2. Consider the spectral problem

$$(-1)^n D_x^{2n} w(x) + A^{2n} w(x) = \lambda w(x), \quad (50)$$

$$\ell_j w \equiv D_x^{2j-1} w(0) - D_x^{2j-1} w(1) + l_j^1 w = 0, \quad (51)$$

$$\ell_{n+j} w \equiv D_x^{2j-2} w(0) - D_x^{2j-2} w(1) = 0, \quad (52)$$

where

$$\ell_j^1 w \equiv \sum_{r=0}^{m_j} (b_{j,r,0} D_x^r w(0) + b_{j,r,1} D_x^r w(1)), \quad j = 1, 2, \dots, n. \quad (53)$$

Let L_3 be the operator of the problem (50)–(53) and

$$L_3 u \equiv (-1)^n D_x^{2n} u + A^{2n} u, \quad u \in D(L_3), \quad D(L_3) \equiv \{u \in H_2 : l_j u = 0, j = 1, 2, \dots, 2n\}.$$

We find the solution of the spectral problem (50)–(53) as the product $w(x) = y(x)v_k$, $v_k \in V(A)$, $k = 1, 2, \dots$.

To determine the unknown function $y \in W^{2n}(0, 1)$ we obtain the spectral problem

$$(-1)^n y^{(2n)} + z_k^{2n} y = \lambda y, \quad \lambda \in \mathbb{C}, \quad (54)$$

$$\ell_j y \equiv y^{(2j-1)}(0) + y^{(2j-1)}(1) + l_j^1 y = 0, \quad (55)$$

$$\ell_{n+j} y \equiv y^{(2j-2)}(0) - y^{(2j-2)}(1) = 0, \quad j = 1, 2, \dots, n. \quad (56)$$

Let $L_{3,k}$ be the operator of the problem (54)–(56);

$$L_{3,k} y(x) \equiv (-1)^n y^{(2n)}(x) + z_k^{2n};$$

$$y \in D(L_{3,k}); \quad D(L_{3,k}) \equiv \left\{ y \in W^{2n}(0, 1) : l_j y = 0, j = 1, 2, \dots, 2n \right\};$$

let $V(L_{3,k})$ be the system of root functions of the operator $L_{3,k}$.

We can prove that

$$v_{2q}(x) \in D(L_{3,k}), \quad L_{3,k} v_{2q}(x) = \lambda_{q,k} v_{2q}(x), \quad q = 0, 1, \dots$$

Therefore,

$$v_{2q}(x, L_{3,k}) \equiv v_{2q}(x), \quad q = 0, 1, \dots \quad (57)$$

The root functions of the operator $L_{3,k}$ are determined by the equalities

$$v_{2q-1}(x, L_{3,k}) \equiv v_{2q-1}(x) + \sum_{p=1}^n c_{1,q,p} y_{2,p}(x, \rho_q), \quad q = 1, 2, \dots \quad (58)$$

Substituting the expression (58) into the boundary conditions (55), (56), we obtain

$$c_{1,q,p} = -\sqrt{2} \sum_{r=0}^{m_p} (-1)^{r-2p+1} b_{p,r,0} (\rho_q)^{2+r-2p}, \quad p = 1, 2, \dots, n, \quad q = 1, 2, \dots \quad (59)$$

Thus, the operator $L_{3,k}$ has the system of root functions (57)–(59) in the means of equalities

$$L_{3,k} v_{2q-1}(x, L_{3,k}) = \lambda_{q,k} v_{2q-1}(x, L_{3,k}) + \zeta_q^0 v_{2q}(x, L_{3,k}),$$

$$\zeta_q^0 = 2\sqrt{2}n (\rho_q)^{2n-1} \sum_{p=1}^n c_{p,q}, \quad q = 1, 2, \dots,$$

$$L_{3,k} v_{2k}(x, L_{3,k}) = \lambda_{q,k} v_{2k}(x, L_{3,k}), \quad q = 0, 1, \dots$$

Let $R(L_{3,k})$ be the operator which acts as $V(L_{0,k}) \rightarrow V(L_{3,k})$. From the formulas (37), (58), we obtain the relation

$$R(L_{3,k}) = \prod_{p=1}^n R(L_{2,k,p}) = E + \sum_{p=1}^n S(L_{2,k,p}). \quad (60)$$

Therefore, we have the inclusion $R(L_{3,k}) \in G_0(L_{0,k}) \subset Q_0(I)$. Thus, we obtain the following statement.

Lemma 5.1. *Suppose that the Assumption P_1 holds. Then, for the any fixed $b_{p,r,0} \in \mathbb{R}$, $r = 0, 1, \dots, m_p$, $p = 1, 2, \dots, n$, the system $V(L_{3,k})$ is complete and minimal in the space H_0 .*

Consider the system V_2 of functions

$$v_{0,2}(x) \equiv v_0(x), \quad v_{2q,2}(x) \equiv v_{2q}(x), \quad q = 1, 2, \dots,$$

$$v_{2q-1,2}(x) \equiv (1 + \tau_2)(1 - 2x)v_{2q-1}(x),$$

$$\tau_2 \equiv W^n (W^{n-1})^{-1} c_b, \quad c_b \equiv \sum_{p=1}^n b_{p,2p-1,0}.$$

Let $R_2 = E + S_2$ be the operator which acts as $V(L_{0,k}) \rightarrow V_2$.

Using that $S_2 : H_{0,1} \rightarrow 0$, $S_2 : H_{0,0} \rightarrow H_0$, we obtain that $R_2 \in Q_0(I)$.

Lemma 5.2. *Suppose that the Assumptions P_1, P_2 hold. Then the system V_2 forms the Riesz basis in the space H_0 .*

The proof is carried out analogously in Lemma 3.1.

Remark 5.3. *Suppose that the Assumptions P_1, P_2 hold. Then the following relations hold*

$$v_{2q-1}(x, L_{3,k}) = v_{2q-1,2}(x) + \sum_{j=2}^n c_{j,q}^1 y_j(x, \rho_q) + (\rho_q)^{-1} c_q^2 (1 - 2x) v_{2q-1}(x), \quad (61)$$

where

$$|c_{j,q}^1| \leq K_{12}, \quad |c_q^2| \leq K_{12} < \infty, \quad q = 0, 1, \dots$$

Therefore, the systems $V(L_{3,k})$ and V_2 are squarely close in the space H_0 .

Lemma 5.3. *Suppose that the Assumptions P_1, P_2 hold. Then, for any fixed $b_{p,r,0} \in \mathbb{R}$, $r = 0, 1, \dots, m_p$, $p = 1, 2, \dots, n$, the system $V(L_{3,k})$ forms the Riesz basis in the space H_0 .*

The statement follows from Lemma 5.1, Lemma 5.2, Remark 5.4 and the Bari Theorem (see [9]).

Theorem 7. *Suppose that the Assumption P_1 holds. Then, for any $b_{p,r,0} \in \mathbb{R}$, $r = 0, 1, \dots, m_p$, $p = 1, 2, \dots, n$, the following assertions are true*

- 1) *the eigenvalues of the operators $L_{0,k}$ and $L_{3,k}$ coincide;*
- 2) *the system $V(L_{3,k})$ is complete and minimal in the space H_0 ;*
- 3) *if in addition the Assumption P_2 holds, then the system $V(L_{3,k})$ forms the Riesz basis of the space H_0 .*

Proof. The proof of part 1 of the theorem follows from Theorem 3, the second statement follows from Lemma 5.1 and the third statement follows from Lemma 5.3. \square

Let

$$V(L_3) \equiv \left\{ v_{q,k}(x, L_3) \equiv v_q(x, L_{3,k})v_k \in W_1 : q = \overline{0, \infty}, k = \overline{1, \infty} \right\}$$

forms the system of the root functions of the operator L_3 . Let

$$W(L_{3,k}) \equiv \left\{ w_{q,k}(x, L_3) \in H_1 : q = \overline{0, \infty} \right\}$$

be the biorthogonal system of functions to the system $V(L_{3,k})$ in the space H_0 . Let

$$W(L_3) \equiv \left\{ w_{q,k}(x, L_3) \equiv w_q(x, L_{3,k})w_k \in H_1 : q = \overline{0, \infty}, k = \overline{1, \infty} \right\},$$

be the biorthogonal system of functions to the system $V(L_3)$ in the space W_1 and $R(L_3)$ be the operator which acts as $V(L_0) \rightarrow V(L_3)$.

Theorem 8. *Suppose that the Assumption P_1 holds. Then, for all numbers $b_{p,r,0} \in \mathbb{R}$, $r = 0, 1, \dots, m_p$, $p = 1, 2, \dots, n$, the following assertions are true*

- 1) *the eigenvalues of the operators L_0 and L_3 coincide;*
- 2) *the system $V(L_3)$ is complete and minimal in the space H_1 ;*
- 3) *if in addition the Assumption P_2 holds, then the system $V(L_3)$ forms the Riesz basis of the space H_1 .*

Proof. The proof of part 1 and 2 of the theorem follows from Theorem 7. Taking the relations (60), (61) into account we obtain the equality

$$R(L_3) \equiv \prod_{j=1}^n R(L_{2,j}) \equiv \prod_{j=1}^n (E + S(L_{2,j})) = E + \sum_{j=1}^n S(L_{2,j}). \quad (62)$$

Let Assumptions P_1 and P_2 hold. Then from the equality (62) and the assertion 3 of Theorem 7 we obtain

$$R(L_3) \in [H_1], \quad R(L_3)^{-1} = E - S(L_3) \in [H_1].$$

Therefore, the system $V(L_3)$ forms the Riesz basis of the space H_1 . \square

Remark 5.4. *The operator L_3 has the system of root functions in the means of the equalities*

$$\begin{aligned} L_3 v_{2q-1,k}(x, L_3) &= \lambda_{q,k} v_{2q-1,k}(x, L_3) + \zeta_{q,k}^0 v_{2q,k}(x, L_3), \\ \zeta_{q,k}^0 &= 2\sqrt{2}\tau_2 n (\rho_q)^{2n-1} \sum_{p=1}^n c_{p,q}, \quad q = 1, 2, \dots, \\ L_3 v_{2q,k}(x, L_3) &= \lambda_{q,k} v_{2q-1,k}(x, L_3), \quad q = 0, 1, \dots \end{aligned}$$

We consider the system of functions

$$V_3 \equiv \left\{ v_{q,k,3}(x) \in H_1 : v_{q,k,3}(x) \equiv v_{q,1}(x)v_k, \quad q = \overline{0, \infty}, \quad k = \overline{1, \infty} \right\}.$$

Remark 5.5. The systems $V(L_3)$ and V_3 are squarely close in the space H_1 .

6 THE SPECTRAL BOUNDARY VALUE PROBLEM FOR A DIFFERENTIAL-OPERATOR EQUATION WITH INVOLUTION

Consider the spectral problem

$$(-1)^n D_x^{2n} u + A^{2n} u + \sum_{s=1}^n a_s \left(D_x^{2s-1} u(x) + D_x^{2s-1} u(1-x) \right) = \lambda u, \quad (63)$$

$$\ell_j u \equiv D_x^{2j-1} u(0) - D_x^{2j-1} u(1) + l_j^1 u = 0, \quad (64)$$

$$\ell_{n+j} u \equiv D_x^{2j-2} u(0) - D_x^{2j-2} u(1) = 0, \quad (65)$$

with

$$\ell_j^1 u \equiv \sum_{r=0}^{m_j} (b_{j,r,0} D_x^r u(0) + b_{j,r,1} D_x^r u(1)), \quad j = 1, 2, \dots, n. \quad (66)$$

Let L be the operator of the problem (63)–(66) and

$$Lu \equiv (-1)^n D_x^{2n} u + A^{2n} u + \sum_{s=1}^n a_s \left(D_x^{2s-1} u(x) + D_x^{2s-1} u(1-x) \right);$$

$$u \in D(L), \quad D(L) \equiv \{u \in H_2 : l_j u = 0, \quad j = 1, 2, \dots, 2n\}.$$

We can prove that

$$Lv_{2q-1,k}(x, L_3) = \lambda_{q,k} v_{2q-1,k}(x, L_3) + \xi_{q,k}^1 v_{2q,k}(x, L_3), \quad q = 1, 2, \dots,$$

$$\xi_{q,k}^1 = \xi_{q,k}^0 + 2\sqrt{2}\tau_2 \sum_{j=1}^n (-1)^{j-1} a_j (\rho_q)^{2j-1} (-4j+2), \quad q = 1, 2, \dots,$$

$$Lv_{2q,k}(x, L_3) = \lambda_{q,k} v_{2q-1,k}(x, L_3), \quad q = 0, 1, \dots.$$

Consequently, $V(L) \equiv V(L_3)$ and the following theorem is true.

Theorem 9. Suppose that the Assumption P_1 holds. Then for the any numbers $b_{p,r,0}$, $a_j \in \mathbb{R}$, $r = 0, 1, \dots, m_p$, j , $p = 1, 2, \dots, n$ we have 1) the eigenvalues of the operators L_0 and L coincide;

2) the system $V(L)$ of the root functions of the operator L is complete and minimal in the space H_1 .

3) if in addition the Assumption P_2 holds, then the system $V(L)$ forms the Riesz basis in the space H_1 .

Let

$$f = \sum_{k=1}^{\infty} \sum_{q=0}^{\infty} f_{q,k} v_{q,k}(x, L), \quad f_{q,k} = (f, w_{q,k}(x, L); H_1).$$

Remark 6.1. From the definition of the Riesz basis of Hilbert space and the third statement of Theorem 9 for any $f \in H_1$ we obtain the relation

$$K_{13} \|f; H_1\|^2 \leq \sum_{k=1}^{\infty} \sum_{q=0}^{\infty} |f_{q,k}|^2 \leq K_{14} \|f; H_1\|^2. \quad (67)$$

7 THE BOUNDARY VALUE PROBLEM WITH HOMOGENEOUS CONDITIONS WITH INVOLUTION

We consider the following boundary problem

$$(-1)^n D_x^{2n} w + A^{2n} w + \sum_{s=1}^n a_s \left(D_x^{2s-1} w(x) + D_x^{2s-1} w(1-x) \right) = f, \quad (68)$$

$$\ell_j w \equiv D_x^{2j-1} w(0) - D_x^{2j-1} w(1) + l_j^1 w = 0, \quad (69)$$

$$\ell_{n+j} w \equiv D_x^{2j-2} w(0) - D_x^{2j-2} w(1) = 0, \quad (70)$$

where

$$\ell_j^1 w \equiv \sum_{r=0}^{m_j} (b_{j,r,0} D_x^r w(0) + b_{j,r,1} D_x^r w(1)), \quad j = 1, 2, \dots, n. \quad (71)$$

Let $W(L)$ be a biorthogonal system of functions from $V(L)$ in the space H_1 .

Theorem 10. *Suppose that the Assumption P_1 holds. Then for arbitrary numbers $b_{p,r,0}, a_j \in \mathbb{R}$, $r = 0, 1, \dots, m_p$, $j, p \in \{1, 2, \dots, n\}$, and function $f \in H_1$ there exists a unique solution of the problem (68)–(71).*

Proof. The solution of the problem (68)–(71) can be determined by the relation

$$w = \sum_{k=1}^{\infty} \sum_{q=0}^{\infty} w_{q,k} v_{q,k}(x, L). \quad (72)$$

Substituting the relations (67), (72) into the equation (68) we obtain

$$w_{2q-1,k} = \lambda_{q,k}^{-1} f_{2q-1,k}, \quad w_{2q,k} = \lambda_{q,k}^{-1} f_{2q,k} - \lambda_{q,k}^{-2} \zeta_{q,k}^1 f_{2q-1,k}, \quad q, k = 1, 2, \dots$$

Therefore,

$$|w_{2q-1,k}|^2 \leq K_{15} |f_{2q-1,k}|^2, \quad (73)$$

$$|w_{2q,k}|^2 \leq K_{16} (|f_{2q-1,k}|^2 + |f_{2q,k}|^2), \quad q, k = 1, 2, \dots \quad (74)$$

Taking the assumption $f \in H_1$ and the inequalities (67) into account we obtain that $\|w; H_1\| \leq K_{17} \|f; H_1\|$, $w \in H_1$.

Consider the function $h_1 \equiv A^{2n} w$

$$h_1 = \sum_{k=1}^{\infty} (f_{0,k} v_{0,k}(x, L) + \sum_{q=1}^{\infty} (z_k^{2n} \lambda_{q,k}^{-1} f_{2q-1,k} v_{2q-1,k}(x, L) + (z_k^{2n} \lambda_{q,k}^{-1} f_{2q,k} - \lambda_{q,k}^{-2} z_k^{2n} \zeta_{q,k}^1 f_{2q-1,k}) v_{2q,k}(x, L))). \quad (75)$$

Taking the assumption $f \in H_1$ and the inequalities (67) into account we obtain that $\|h_1; H_1\| \leq K_{18} \|f; H_1\|$, $h_1 \in H_1$.

Consider the function $h_2 \equiv (-1)^n D_2^{2n} w$

$$h_2 = \sum_{k=1}^{\infty} \sum_{q=1}^{\infty} (\rho_q^{2n} \lambda_{q,k}^{-1} f_{2q-1,k} v_{2q-1,k}(x, L) + (\rho_q^{2n} \lambda_{q,k}^{-1} f_{2q,k} - 2\rho_q^{2n} \lambda_{q,k}^{-2} \zeta_{q,k}^1 f_{2q-1,k}) v_{2q,k}(x, L)). \quad (76)$$

From the equalities (76) we get $\|h_2; H_1\| \leq K_{19} \|f; H_1\|$, $h_2 \in H_1$.

Consider the function $h_3 = \sum_{s=1}^n a_s (D_x^{2s-1} w(x) - D_x^{2s-1} w(1-x))$

$$h_3 = 2 \sum_{k=1}^{\infty} \sum_{q=1}^{\infty} \sum_{j=1}^n a_j (-1)^{j-1} \rho_q^{2j-1} (\lambda_{q,k}^{-1} f_{2q,k} - \lambda_{q,k}^{-2} \alpha_{q,k}^1 f_{2q-1,k}) v_{2q-1,k}(x, L). \quad (77)$$

Taking the assumption $f \in H_1$ and the equalities (77) into account we obtain that

$$\|h_3; H_1\| \leq K_{20} \|f; H_1\|.$$

From the definition of the space H_2 , inequalities (75)–(77) and Cauchy's inequality we get

$$\|w; H_2\| \leq K_{21} \|f; H_1\| < \infty, \quad K_{21} = 3(\max(K_{17}^2, K_{18}^2, K_{19}^2))^{\frac{1}{2}}.$$

Thus $w \in H_2$. □

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Баранецький Я.О., Демків І.І., Івасюк І.Я., Копач М.І. *Нелокальна задача для диференціально-операторного рівняння порядку $2n$ з необмеженими операторними коефіцієнтами з інволюцією* // Карпатські матем. публ. — 2018. — Т.10, №1. — С. 14–30.

Вивчається нелокальна крайова задача для диференціально-операторного рівняння парного порядку, який містить оператор інволюції. Досліджується задача з періодичними крайовими умовами для диференціального рівняння, коефіцієнти якого є несамоспрженими операторами. Встановлено, що оператор задачі має два інваріантні підпростори, породжені оператором інволюції та дві підсистеми системи власних функцій, які є базисами Рісса в кожному з підпросторів. Для диференціально-операторного рівняння парного порядку вивчається задача з несамоспрженими крайовими умовами, які є збуреннями періодичних умов. Вивчено випадки, коли збурені умови є регулярними, але не сильно регулярними за Біркгофом та нерегулярними за Біркгофом. Визначено власні значення і елементи системи кореневих функцій V оператора задачі, яка є повною та містить нескінченне число приєднаних функцій. Отримано достатні умови, при яких система V є базисом Рісса. Визначено умови існування та єдиності розв'язку задачі з однорідними крайовими умовами, який побудовано у вигляді ряду за системою V .

Ключові слова і фрази: оператор інволюції, диференціально-операторне рівняння, власні функції, базис Рісса.



BISWAS T.

ADVANCEMENT ON THE STUDY OF GROWTH ANALYSIS OF DIFFERENTIAL POLYNOMIAL AND DIFFERENTIAL MONOMIAL IN THE LIGHT OF SLOWLY INCREASING FUNCTIONS

Study of the growth analysis of entire or meromorphic functions has generally been done through their Nevanlinna's characteristic function in comparison with those exponential functions. But if one is interested to compare the growth rates of any entire or meromorphic function with respect to another, the concepts of relative growth indicators will come. The field of study in this area may be more significant through the intensive applications of the theories of slowly increasing functions which actually means that $L(ar) \sim L(r)$ as $r \rightarrow \infty$ for every positive constant a , i.e. $\lim_{r \rightarrow \infty} \frac{L(ar)}{L(r)} = 1$, where $L \equiv L(r)$ is a positive continuous function increasing slowly. Actually in the present paper, we establish some results depending on the comparative growth properties of composite entire and meromorphic functions using the idea of relative pL^* -order, relative pL^* -type, relative pL^* -weak type and differential monomials, differential polynomials generated by one of the factors which extend some earlier results, where pL^* is nothing but a weaker assumption of L .

Key words and phrases: entire function, meromorphic function, relative pL^* -order, relative pL^* -type, relative pL^* -weak type, growth, differential monomial, differential polynomial, slowly increasing function.

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INTRODUCTION, DEFINITIONS AND NOTATIONS

Let us consider that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna theory of meromorphic functions which are available in [13, 16, 22, 23]. We also use the standard notations and definitions of the theory of entire functions which are available in [24] and therefore we do not explain those in details.

For $x \in [0, \infty)$ and $k \in \mathbb{N}$, we define the following functions $\exp^{[k]} x = \exp(\exp^{[k-1]} x)$ and $\log^{[k]} x = \log(\log^{[k-1]} x)$, where \mathbb{N} be the set of all positive integers.

Let f be an entire function defined in the open complex plane \mathbb{C} . The maximum modulus function $M_f(r)$ corresponding to f is defined on $|z| = r$ as $M_f(r) = \max_{|z|=r} |f(z)|$. In this connection the following definition is relevant.

Definition 1 ([4]). *A non-constant entire function f is said have the Property (A) if for any $\sigma > 1$ and for all sufficiently large r , $[M_f(r)]^2 \leq M_f(r^\sigma)$ holds.*

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For examples of functions with or without the Property (A) we refer the reader to [4].

When f is meromorphic, one may introduce another function $T_f(r)$ known as Nevanlinna's characteristic function of f , playing the same role as $M_f(r)$.

Now we just recall the following properties of meromorphic functions which will be needed in the sequel.

Let $n_{0j}, n_{1j}, \dots, n_{kj}$ ($k \geq 1$) be non-negative integers such that for each j the following inequality holds $\sum_{i=0}^k n_{ij} \geq 1$. For a non-constant meromorphic function f , we call $M_j[f] = A_j(f)^{n_{0j}} (f^{(1)})^{n_{1j}} \dots (f^{(k)})^{n_{kj}}$, where $T(r, A_j) = S(r, f)$ to be a differential monomial generated by f . The numbers $\gamma_{M_j} = \sum_{i=0}^k n_{ij}$ and $\Gamma_{M_j} = \sum_{i=0}^k (i+1)n_{ij}$ are called the degree and weight of $M_j[f]$ respectively [6, 19]. The expression $P[f] = \sum_{j=1}^s M_j[f]$ is called a differential polynomial generated by f . The numbers $\gamma_P = \max_{1 \leq j \leq s} \gamma_{M_j}$ and $\Gamma_P = \max_{1 \leq j \leq s} \Gamma_{M_j}$ are called the degree and weight of $P[f]$ respectively [6, 19]. Also we call the numbers $\gamma_P = \min_{1 \leq j \leq s} \gamma_{M_j}$ and k (the order of the highest derivative of f) the lower degree and the order of $P[f]$ respectively. If $\gamma_P = \gamma_P$, $P[f]$ is called a homogeneous differential polynomial. Throughout the paper, we consider only the non-constant differential polynomials and we denote by $P_0[f]$ a differential polynomial not containing f , i.e. for which $n_{0j} = 0$ for $j = 1, 2, \dots, s$. We consider only those $P[f], P_0[f]$ singularities of whose individual terms do not cancel each other. We also denote by $M[f]$ a differential monomial generated by a transcendental meromorphic function f .

However, the Nevanlinna's Characteristic function of a meromorphic function f is defined as

$$T_f(r) = N_f(r) + m_f(r),$$

wherever the function $N_f(r, a)$ ($\bar{N}_f(r, a)$) known as counting function of a -points (distinct a -points) of meromorphic f is defined as follows:

$$N_f(r, a) = \int_0^r \frac{n_f(t, a) - n_f(0, a)}{t} dt + n_f(0, a) \log r$$

$$\left(\bar{N}_f(r, a) = \int_0^r \frac{\bar{n}_f(t, a) - \bar{n}_f(0, a)}{t} dt + \bar{n}_f(0, a) \log r \right),$$

in addition we represent by $n_f(r, a)$ ($\bar{n}_f(r, a)$) the number of a -points (distinct a -points) of f in $|z| \leq r$ and an ∞ -point is a pole of f . In many occasions $N_f(r, \infty)$ and $\bar{N}_f(r, \infty)$ are symbolized by $N_f(r)$ and $\bar{N}_f(r)$ respectively.

On the other hand, the function $m_f(r, \infty)$ alternatively indicated by $m_f(r)$ known as the proximity function of f is defined as follows

$$m_f(r) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta, \text{ where } \log^+ x = \max(\log x, 0) \text{ for all } x \geq 0.$$

Also we may employ $m\left(r, \frac{1}{f-a}\right)$ by $m_f(r, a)$.

If f is entire, then the Nevanlinna's Characteristic function $T_f(r)$ of f is defined as

$$T_f(r) = m_f(r).$$

Moreover for any non-constant entire function f , $T_f(r)$ is strictly increasing and continuous functions of r . Also its inverse $T_f^{-1} : (|T_f(0)|, \infty) \rightarrow (0, \infty)$ exists, where $\lim_{s \rightarrow \infty} T_f^{-1}(s) = \infty$.

In this connection we immediately remind the following definition which is relevant.

Definition 2. Let a be a complex number, finite or infinite. The Nevanlinna's deficiency and the Valiron deficiency of a with respect to a meromorphic function f are defined as

$$\delta(a; f) = 1 - \overline{\lim}_{r \rightarrow \infty} \frac{N_f(r, a)}{T_f(r)} = \underline{\lim}_{r \rightarrow \infty} \frac{m_f(r, a)}{T_f(r)}$$

and

$$\Delta(a; f) = 1 - \underline{\lim}_{r \rightarrow \infty} \frac{N_f(r, a)}{T_f(r)} = \overline{\lim}_{r \rightarrow \infty} \frac{m_f(r, a)}{T_f(r)}.$$

Definition 3. The quantity $\Theta(a; f)$ of a meromorphic function f is defined as follows

$$\Theta(a; f) = 1 - \overline{\lim}_{r \rightarrow \infty} \frac{\bar{N}_f(r, a)}{T_f(r)}.$$

Definition 4 ([21]). For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $n_{f|=1}(r, a)$, the number of simple zeros of $f - a$ in $|z| \leq r$. $N_{f|=1}(r, a)$ is defined in terms of $n_{f|=1}(r, a)$ in the usual way. We put

$$\delta_1(a; f) = 1 - \overline{\lim}_{r \rightarrow \infty} \frac{N_{f|=1}(r, a)}{T_f(r)},$$

the deficiency of a corresponding to the simple a -points of f , i.e. simple zeros of $f - a$.

Yang [20] proved that there exists at most a denumerable number of complex numbers $a \in \mathbb{C} \cup \{\infty\}$ for which $\delta_1(a; f) > 0$ and $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; f) \leq 4$.

Definition 5 ([14]). For $a \in \mathbb{C} \cup \{\infty\}$, let $n_p(r, a; f)$ denotes the number of zeros of $f - a$ in $|z| \leq r$, where a zero of multiplicity $< p$ is counted according to its multiplicity and a zero of multiplicity $\geq p$ is counted exactly p times and $N_p(r, a; f)$ is defined in terms of $n_p(r, a; f)$ in the usual way. We define

$$\delta_p(a; f) = 1 - \overline{\lim}_{r \rightarrow \infty} \frac{N_p(r, a; f)}{T_f(r)}.$$

Definition 6 ([1]). $P[f]$ is said to be admissible if

- (i) $P[f]$ is homogeneous, or
- (ii) $P[f]$ is non homogeneous and $m_f(r) = S_f(r)$.

However in case of any two meromorphic functions f and g , the ratio $\frac{T_f(r)}{T_g(r)}$ as $r \rightarrow \infty$ is called as the growth of f with respect to g in terms of their Nevanlinna's Characteristic functions. Further the concept of the growth measuring tools such as order and lower order which are conventional in complex analysis and the growth of entire or meromorphic functions can be studied in terms of their orders and lower orders are normally defined in terms of their growth with respect to the exp function which are shown in the following definition.

Definition 7. The order ρ_f (the lower order λ_f) of a meromorphic function f is defined as

$$\rho_f = \overline{\lim}_{r \rightarrow \infty} \frac{\log T_f(r)}{\log T_{\exp z}(r)} = \overline{\lim}_{r \rightarrow \infty} \frac{\log T_f(r)}{\log \left(\frac{r}{\pi}\right)} = \overline{\lim}_{r \rightarrow \infty} \frac{\log T_f(r)}{\log(r) + O(1)}$$

$$\left(\lambda_f = \underline{\lim}_{r \rightarrow \infty} \frac{\log T_f(r)}{\log T_{\exp z}(r)} = \underline{\lim}_{r \rightarrow \infty} \frac{\log T_f(r)}{\log \left(\frac{r}{\pi}\right)} = \underline{\lim}_{r \rightarrow \infty} \frac{\log T_f(r)}{\log(r) + O(1)} \right).$$

If f is entire, then

$$\rho_f = \overline{\lim}_{r \rightarrow \infty} \frac{\log \log M_f(r)}{\log \log M_{\exp z}(r)} = \overline{\lim}_{r \rightarrow \infty} \frac{\log^{[2]} M_f(r)}{\log r}$$

$$\left(\lambda_f = \underline{\lim}_{r \rightarrow \infty} \frac{\log \log M_f(r)}{\log \log M_{\exp z}(r)} = \underline{\lim}_{r \rightarrow \infty} \frac{\log^{[2]} M_f(r)}{\log r} \right).$$

Somasundaram and Thamizharasi [18] introduced the notions of L -order and L -type for entire functions, where $L \equiv L(r)$ is a positive continuous function increasing slowly, i.e. $L(ar) \sim L(r)$ as $r \rightarrow \infty$ for every positive constant a . The more generalized concept of L -order and L -type of meromorphic functions are L^* -order and L^* -type (resp. L^* -lower type) respectively which are as follows.

Definition 8 ([18]). The L^* -order $\rho_f^{L^*}$ and the L^* -lower order $\lambda_f^{L^*}$ of a meromorphic function f are defined by

$$\rho_f^{L^*} = \overline{\lim}_{r \rightarrow \infty} \frac{\log T_f(r)}{\log [re^{L(r)}]} \quad \text{and} \quad \lambda_f^{L^*} = \underline{\lim}_{r \rightarrow \infty} \frac{\log T_f(r)}{\log [re^{L(r)}]},$$

where $L \equiv L(r)$ is a positive continuous function increasing slowly.

If f is entire, then

$$\rho_f^{L^*} = \overline{\lim}_{r \rightarrow \infty} \frac{\log^{[2]} M_f(r)}{\log [re^{L(r)}]} \quad \text{and} \quad \lambda_f^{L^*} = \underline{\lim}_{r \rightarrow \infty} \frac{\log^{[2]} M_f(r)}{\log [re^{L(r)}]}.$$

Definition 9 ([18]). The L^* -type $\sigma_f^{L^*}$ and L^* -lower type $\bar{\sigma}_f^{L^*}$ of a meromorphic function f such that $0 < \rho_f^{L^*} < \infty$ are defined as

$$\sigma_f^{L^*} = \overline{\lim}_{r \rightarrow \infty} \frac{T_f(r)}{[re^{L(r)}]^{\rho_f^{L^*}}} \quad \text{and} \quad \bar{\sigma}_f^{L^*} = \underline{\lim}_{r \rightarrow \infty} \frac{T_f(r)}{[re^{L(r)}]^{\rho_f^{L^*}}},$$

where $L \equiv L(r)$ is a positive continuous function increasing slowly.

If f is entire, then

$$\sigma_f^{L^*} = \overline{\lim}_{r \rightarrow \infty} \frac{\log M_f(r)}{[re^{L(r)}]^{\rho_f^{L^*}}} \quad \text{and} \quad \bar{\sigma}_f^{L^*} = \underline{\lim}_{r \rightarrow \infty} \frac{\log M_f(r)}{[re^{L(r)}]^{\rho_f^{L^*}}}.$$

Analogously in order to determine the relative growth of two meromorphic functions having same non zero finite L^* -lower order one may introduce the definition of L^* -weak type of meromorphic functions having finite positive L^* -lower order in the following way.

Definition 10. The L^* -weak type denoted by $\tau_f^{L^*}$ of a meromorphic function f having $0 < \lambda_f^{L^*} < \infty$ is defined as follows

$$\tau_f^{L^*} = \liminf_{r \rightarrow \infty} \frac{T_f(r)}{[re^{L(r)}]_{\lambda_f^{L^*}}},$$

where $L \equiv L(r)$ is a positive continuous function increasing slowly.

Similarly the growth indicator $\bar{\tau}_f^{L^*}$ is defined as

$$\bar{\tau}_f^{L^*} = \overline{\lim}_{r \rightarrow \infty} \frac{T_f(r)}{[re^{L(r)}]_{\lambda_f^{L^*}}}, \quad \text{where } 0 < \lambda_f^{L^*} < \infty.$$

If f is entire, then

$$\tau_f^{L^*} = \liminf_{r \rightarrow \infty} \frac{\log M_f(r)}{[re^{L(r)}]_{\lambda_f^{L^*}}} \quad \text{and} \quad \bar{\tau}_f^{L^*} = \overline{\lim}_{r \rightarrow \infty} \frac{\log M_f(r)}{[re^{L(r)}]_{\lambda_f^{L^*}}}, \quad \text{where } 0 < \lambda_f^{L^*} < \infty.$$

Extending the notion of Somasundaram and Thamizharasi [18], one may introduce concept of pL^* -order, pL^* -type and pL^* -weak type of a meromorphic function f as follows.

Definition 11. For any positive integer p , the pL^* -order $\rho_p^{L^*}(f)$ and the pL^* -lower order $\lambda_p^{L^*}(f)$ of a meromorphic function f are defined by

$$\rho_p^{L^*}(f) = \overline{\lim}_{r \rightarrow \infty} \frac{\log T_f(r)}{\log [r \exp^{[p]} L(r)]} \quad \text{and} \quad \lambda_p^{L^*}(f) = \liminf_{r \rightarrow \infty} \frac{\log T_f(r)}{\log [r \exp^{[p]} L(r)]},$$

where $L \equiv L(r)$ is a positive continuous function increasing slowly.

If f is entire, then

$$\rho_p^{L^*}(f) = \overline{\lim}_{r \rightarrow \infty} \frac{\log^{[2]} M_f(r)}{\log [r \exp^{[p]} L(r)]} \quad \text{and} \quad \lambda_p^{L^*}(f) = \liminf_{r \rightarrow \infty} \frac{\log^{[2]} M_f(r)}{\log [r \exp^{[p]} L(r)]}.$$

Definition 12. For any positive integer p , the pL^* -type $\sigma_p^{L^*}(f)$ and pL^* -lower type $\bar{\sigma}_p^{L^*}(f)$ of a meromorphic function f such that $0 < \rho_p^{L^*}(f) < \infty$ are defined by

$$\sigma_p^{L^*}(f) = \overline{\lim}_{r \rightarrow \infty} \frac{T_f(r)}{[r \exp^{[p]} L(r)]^{\rho_p^{L^*}(f)}} \quad \text{and} \quad \bar{\sigma}_p^{L^*}(f) = \liminf_{r \rightarrow \infty} \frac{T_f(r)}{[r \exp^{[p]} L(r)]^{\rho_p^{L^*}(f)}},$$

where $L \equiv L(r)$ is a positive continuous function increasing slowly.

If f is entire, then

$$\sigma_p^{L^*}(f) = \overline{\lim}_{r \rightarrow \infty} \frac{\log M_f(r)}{[r \exp^{[p]} L(r)]^{\rho_p^{L^*}(f)}} \quad \text{and} \quad \bar{\sigma}_p^{L^*}(f) = \liminf_{r \rightarrow \infty} \frac{\log M_f(r)}{[r \exp^{[p]} L(r)]^{\rho_p^{L^*}(f)}}.$$

Definition 13. For any positive integer p , the pL^* -weak type denoted by $\tau_p^{L^*}(f)$ of a meromorphic function f having $0 < \lambda_p^{L^*}(f) < \infty$ is defined by

$$\tau_p^{L^*}(f) = \liminf_{r \rightarrow \infty} \frac{T_f(r)}{[r \exp^{[p]} L(r)]^{\lambda_p^{L^*}(f)}},$$

where $L \equiv L(r)$ is a positive continuous function increasing slowly.

Similarly the growth indicator $\bar{\tau}_p^{L^*}(f)$ is defined by

$$\bar{\tau}_p^{L^*}(f) = \overline{\lim}_{r \rightarrow \infty} \frac{T_f(r)}{[r \exp^{[p]} L(r)]^{\lambda_p^{L^*}(f)}}, \quad \text{where } 0 < \lambda_p^{L^*}(f) < \infty.$$

If f is entire, then for $0 < \lambda_p^{L^*}(f) < \infty$,

$$\tau_p^{L^*}(f) = \underline{\lim}_{r \rightarrow \infty} \frac{\log M_f(r)}{[r \exp^{[p]} L(r)]^{\lambda_p^{L^*}(f)}} \quad \text{and} \quad \bar{\tau}_p^{L^*}(f) = \overline{\lim}_{r \rightarrow \infty} \frac{\log M_f(r)}{[r \exp^{[p]} L(r)]^{\lambda_p^{L^*}(f)}}.$$

Lahiri and Banerjee [17] introduced the following definition of relative order of a meromorphic function with respect to an entire function.

Definition 14 ([17]). Let f be meromorphic and g be entire functions. The relative order of f with respect to g denoted by $\rho_g(f)$ is defined as

$$\begin{aligned} \rho(f, g) &= \inf \{ \mu > 0 : T_f(r) < T_g(r^\mu) \text{ for all sufficiently large } r \} \\ &= \underline{\lim}_{r \rightarrow \infty} \frac{\log T_g^{-1} T_f(r)}{\log r}. \end{aligned}$$

The definition coincides with the classical one [17] if $g(z) = \exp z$.

Similarly one can define the relative lower order of a meromorphic function f with respect to an entire g denoted by $\lambda_g(f)$ in the following manner

$$\lambda(f, g) = \underline{\lim}_{r \rightarrow \infty} \frac{\log T_g^{-1} T_f(r)}{\log r}.$$

In order to make some progress in the study of relative order, now we introduce relative $_p L^*$ -order and relative $_p L^*$ -lower order of a meromorphic function f with respect to an entire function g .

Definition 15. The relative $_p L^*$ -order denoted as $\rho_p^{L^*}(f, g)$ and relative $_p L^*$ -lower order denoted as $\lambda_p^{L^*}(f, g)$ of a meromorphic function f with respect to an entire g are defined as

$$\rho_p^{L^*}(f, g) = \overline{\lim}_{r \rightarrow \infty} \frac{\log T_g^{-1} T_f(r)}{\log [r \exp^{[p]} L(r)]} \quad \text{and} \quad \lambda_p^{L^*}(f, g) = \underline{\lim}_{r \rightarrow \infty} \frac{\log T_g^{-1} T_f(r)}{\log [r \exp^{[p]} L(r)]},$$

where p is any positive integers and $L \equiv L(r)$ is a positive continuous function increasing slowly.

Further to compare the relative growth of two meromorphic functions having same non zero finite relative $_p L^*$ -order with respect to another entire function, one may introduce the definitions of relative $_p L^*$ -type and relative $_p L^*$ -lower type in the following manner.

Definition 16. The relative $_p L^*$ -type and relative $_p L^*$ -lower type denoted respectively by $\sigma_p^{L^*}(f, g)$ and $\bar{\sigma}_p^{L^*}(f, g)$ of a meromorphic function f with respect to an entire function g such that $0 < \rho_p^{L^*}(f, g) < \infty$ are respectively defined by

$$\sigma_p^{L^*}(f, g) = \overline{\lim}_{r \rightarrow \infty} \frac{T_g^{-1} T_f(r)}{[r \exp^{[p]} L(r)]^{\rho_p^{L^*}(f, g)}} \quad \text{and} \quad \bar{\sigma}_p^{L^*}(f, g) = \underline{\lim}_{r \rightarrow \infty} \frac{T_g^{-1} T_f(r)}{[r \exp^{[p]} L(r)]^{\rho_p^{L^*}(f, g)}},$$

where $L \equiv L(r)$ is a positive continuous function increasing slowly.

Analogously to determine the relative growth of two meromorphic functions having same non zero finite relative pL^* -lower order with respect to an entire function one may introduce the definition of relative pL^* -weak type in the following way.

Definition 17. The relative pL^* -weak type denoted by $\tau_p^{L^*}(f, g)$ of a meromorphic function f with respect to an entire function g such that $0 < \lambda_p^{L^*}(f, g) < \infty$ is defined by

$$\tau_p^{L^*}(f, g) = \liminf_{r \rightarrow \infty} \frac{T_g^{-1} T_f(r)}{[r \exp^{[p]} L(r)]^{\lambda_p^{L^*}(f, g)}},$$

where $L \equiv L(r)$ is a positive continuous function increasing slowly.

Similarly one may define the growth indicator $\bar{\tau}_p^{L^*}(f, g)$ of a meromorphic function f with respect to an entire function g as follows

$$\bar{\tau}_p^{L^*}(f, g) = \overline{\lim}_{r \rightarrow \infty} \frac{T_g^{-1} T_f(r)}{[r \exp^{[p]} L(r)]^{\lambda_p^{L^*}(f, g)}}, \quad 0 < \lambda_p^{L^*}(f, g) < \infty.$$

In the paper we establish some new results depending on the comparative growth properties of composite entire or meromorphic functions using relative pL^* -order, relative pL^* -type, relative pL^* -weak type and differential monomials, differential polynomials generated by one of the factors which in fact extend and improve some results of [9] and [10].

1 LEMMAS

In this section we present some lemmas which will be needed in the sequel.

Lemma 1 ([7]). Let f be a meromorphic function either of finite order or of non-zero lower order such that $\Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1$ and h be an entire function with regular growth and non zero finite type. Also let $\Theta(\infty; h) = \sum_{a \neq \infty} \delta_p(a; h) = 1$ or $\delta(\infty; h) = \sum_{a \neq \infty} \delta(a; h) = 1$. Then for homogeneous $P_0[f]$ and $P_0[g]$,

$$\lim_{r \rightarrow \infty} \frac{T_{P_0[h]}^{-1} T_{P_0[f]}(r)}{T_h^{-1} T_f(r)} = \left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}} \right)^{\frac{1}{\rho_h}}.$$

Lemma 2 ([8]). Let f be a transcendental meromorphic function of finite order or of non-zero lower order and $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; f) = 4$ and h be a transcendental entire function with regular growth and non zero finite type. Also let $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; h) = 4$. Then

$$\lim_{r \rightarrow \infty} \frac{T_{P[h]}^{-1} T_{P[f]}(r)}{T_h^{-1} T_f(r)} = \left(\frac{\Gamma_{M[f]} - (\Gamma_{M[f]} - \gamma_{M[f]}) \Theta(\infty; f)}{\Gamma_{M[h]} - (\Gamma_{M[h]} - \gamma_{M[h]}) \Theta(\infty; h)} \right)^{\frac{1}{\rho_h}},$$

where

$$\Theta(\infty; f) = 1 - \overline{\lim}_{r \rightarrow \infty} \frac{\bar{N}_f(r)}{T_f(r)} \quad \text{and} \quad \Theta(\infty; h) = 1 - \overline{\lim}_{r \rightarrow \infty} \frac{\bar{N}_h(r)}{T_h(r)}.$$

Lemma 3 ([5]). Let f be a meromorphic function either of finite order or of non-zero lower order such that $\Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1$ and h be an entire function with regular growth having non zero finite order and $\Theta(\infty; h) = \sum_{a \neq \infty} \delta_p(a; h) = 1$ or $\delta(\infty; h) = \sum_{a \neq \infty} \delta(a; h) = 1$. Then for any positive integer p , the relative pL^* -order and relative pL^* -lower order of $P_0[f]$ with respect to $P_0[h]$ are same as those of f with respect to h for homogeneous $P_0[f]$ and $P_0[h]$.

Lemma 4 ([5]). Let f be a transcendental meromorphic function of finite order or of non-zero lower order and $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; f) = 4$ and h be a transcendental entire function with regular growth and non zero finite order. Also let $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; h) = 4$. Then for any positive integer p , the relative pL^* -order and relative pL^* -lower order of $M[f]$ with respect to $M[h]$ are same as those of f with respect to h , i.e.

$$\rho_p^{L^*}(M[f], M[h]) = \rho_p^{L^*}(f, h) \quad \text{and} \quad \lambda_p^{L^*}(M[f], M[h]) = \lambda_p^{L^*}(f, h).$$

Lemma 5. Let f be a meromorphic function either of finite order or of non-zero lower order such that $\Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1$ and h be an entire function of regular growth having non zero finite type and $\Theta(\infty; h) = \sum_{a \neq \infty} \delta_p(a; h) = 1$ or $\delta(\infty; h) = \sum_{a \neq \infty} \delta(a; h) = 1$. Then for any positive integer p , the relative pL^* -type and relative pL^* -lower

type of $P_0[f]$ with respect to $P_0[h]$ are $\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}}$ times that of f with respect to h if $\rho_p^{L^*}(f, h)$ is positive finite, where $P_0[f]$ and $P_0[h]$ are homogeneous.

Proof. By Lemma 3 and Lemma 1 and above we get that

$$\begin{aligned} \sigma_p^{L^*}(P_0[f], P_0[h]) &= \overline{\lim}_{r \rightarrow \infty} \frac{T_{P_0[h]}^{-1} T_{P_0[f]}(r)}{[r \exp^{[p]} L(r)]^{\rho_p^{L^*}(P_0[f], P_0[h])}} \\ &= \lim_{r \rightarrow \infty} \frac{T_{P_0[h]}^{-1} T_{P_0[f]}(r)}{T_h^{-1} T_f(r)} \cdot \overline{\lim}_{r \rightarrow \infty} \frac{T_h^{-1} T_f(r)}{[r \exp^{[p]} L(r)]^{\rho_p^{L^*}(f, h)}} = \left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}} \cdot \sigma_p^{L^*}(f, h). \end{aligned}$$

$$\text{Similarly } \overline{\sigma}_p^{L^*}(P_0[f], P_0[h]) = \left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}} \cdot \overline{\sigma}_p^{L^*}(f, h). \quad \square$$

In the line of Lemma 5 we may state the following lemma without its proof.

Lemma 6. Let f be a meromorphic function either of finite order or of non-zero lower order such that $\Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1$ and h be an entire function of regular growth having non zero finite type and $\Theta(\infty; h) = \sum_{a \neq \infty} \delta_p(a; h) = 1$ or $\delta(\infty; h) =$

$\sum_{a \neq \infty} \delta(a; h) = 1$. Then $\tau_p^{L^*}(P_0[f], P_0[h])$ and $\overline{\tau}_p^{L^*}(P_0[f], P_0[h])$ are $\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}}$ times that of f with respect to h , i.e.

$$\tau_p^{L^*}(P_0[f], P_0[h]) = \left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}} \cdot \tau_p^{L^*}(f, h) \quad \text{and} \quad \overline{\tau}_p^{L^*}(P_0[f], P_0[h]) = \left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}} \cdot \overline{\tau}_p^{L^*}(f, h),$$

when $\lambda_p^{L^*}(f, h)$ is positive finite and $P_0[f], P_0[h]$ are homogeneous.

In the line of Lemma 5 and with the help of Lemma 2 and Lemma 4, we may state the following two lemmas without their proofs.

Lemma 7. Let f be a transcendental meromorphic function of finite order or of non-zero lower order and $\sum_{a \in \text{CU}\{\infty\}} \delta_1(a; f) = 4$ and h be a transcendental entire function of regular growth having non zero finite type and $\sum_{a \in \text{CU}\{\infty\}} \delta_1(a; h) = 4$. Then for any positive integer p , the relative pL^* -type and relative pL^* -lower type of $M[f]$ with respect to $M[h]$ are $\left(\frac{\Gamma_{M[f]} - (\Gamma_{M[f]} - \gamma_{M[f]})\Theta(\infty; f)}{\Gamma_{M[h]} - (\Gamma_{M[h]} - \gamma_{M[h]})\Theta(\infty; h)}\right)^{\frac{1}{\rho_h}}$ times that of f with respect to h if $\rho_p^{L^*}(f, h)$ is positive finite, where

$$\Theta(\infty; f) = 1 - \overline{\lim}_{r \rightarrow \infty} \frac{\overline{N}_f(r)}{T_f(r)} \quad \text{and} \quad \Theta(\infty; h) = 1 - \overline{\lim}_{r \rightarrow \infty} \frac{\overline{N}_h(r)}{T_h(r)}.$$

Lemma 8. Let f be a transcendental meromorphic function of finite order or of non-zero lower order and $\sum_{a \in \text{CU}\{\infty\}} \delta_1(a; f) = 4$ and h be a transcendental entire function of regular growth having non zero finite type and $\sum_{a \in \text{CU}\{\infty\}} \delta_1(a; h) = 4$. Then $\tau_p^{L^*}(M[f], M[h])$ and $\overline{\tau}_p^{L^*}(M[f], M[h])$ are $\left(\frac{\Gamma_{M[f]} - (\Gamma_{M[f]} - \gamma_{M[f]})\Theta(\infty; f)}{\Gamma_{M[h]} - (\Gamma_{M[h]} - \gamma_{M[h]})\Theta(\infty; h)}\right)^{\frac{1}{\rho_h}}$ times that of f with respect to h , i.e.

$$\tau_p^{L^*}(M[f], M[h]) = \left(\frac{\Gamma_{M[f]} - (\Gamma_{M[f]} - \gamma_{M[f]})\Theta(\infty; f)}{\Gamma_{M[h]} - (\Gamma_{M[h]} - \gamma_{M[h]})\Theta(\infty; h)}\right)^{\frac{1}{\rho_h}} \cdot \tau_p^{L^*}(f, h)$$

$$\text{and} \quad \overline{\tau}_p^{L^*}(M[f], M[h]) = \left(\frac{\Gamma_{M[f]} - (\Gamma_{M[f]} - \gamma_{M[f]})\Theta(\infty; f)}{\Gamma_{M[h]} - (\Gamma_{M[h]} - \gamma_{M[h]})\Theta(\infty; h)}\right)^{\frac{1}{\rho_h}} \cdot \overline{\tau}_p^{L^*}(f, h),$$

when $\lambda_p^{L^*}(f, h)$ is positive finite and

$$\Theta(\infty; f) = 1 - \overline{\lim}_{r \rightarrow \infty} \frac{\overline{N}_f(r)}{T_f(r)} \quad \text{and} \quad \Theta(\infty; h) = 1 - \overline{\lim}_{r \rightarrow \infty} \frac{\overline{N}_h(r)}{T_h(r)}.$$

Lemma 9 ([2]). If f is a meromorphic function and g is an entire function then for all sufficiently large values of r we have

$$T_{f \circ g}(r) \leq \{1 + o(1)\} \frac{T_g(r)}{\log M_g(r)} T_f(M_g(r)).$$

Lemma 10 ([3]). Let f be meromorphic function and g be entire function and suppose that $0 < \mu < \rho_g \leq \infty$. Then for a sequence of values of r tending to infinity

$$T_{f \circ g}(r) \geq T_f(\exp(r^\mu)).$$

Lemma 11 ([15]). Let f be meromorphic function and g be entire function such that $0 < \rho_g < \infty$ and $0 < \lambda_f$. Then for a sequence of values of r tending to infinity

$$T_{f \circ g}(r) > T_g(\exp(r^\mu)),$$

where $0 < \mu < \rho_g$.

Lemma 12 ([11]). *Let f be a meromorphic function and g be an entire function such that $\lambda_g < \mu < \infty$ and $0 < \lambda_f \leq \rho_f < \infty$. Then for a sequence of values of r tending to infinity*

$$T_{f \circ g}(r) < T_f(\exp(r^\mu)).$$

Lemma 13 ([11]). *Let f be a meromorphic function of finite order and g be an entire function such that $0 < \lambda_g < \mu < \infty$. Then for a sequence of values of r tending to infinity*

$$T_{f \circ g}(r) < T_g(\exp(r^\mu)).$$

Lemma 14 ([12]). *Let f be an entire function which satisfies the Property (A), $\beta > 0$, $\delta > 1$ and $\alpha > 2$. Then*

$$\beta T_f(r) < T_f(\alpha r^\delta).$$

2 THEOREMS

In this section we present the main results of the paper. It is needless to mention that in the paper, the admissibility and homogeneity of $P_0[f]$ for meromorphic f will be needed as per the requirements of the theorems.

Theorem 1. *Let the meromorphic function f and entire function h satisfy the conditions of Lemma 3. Also let g be an entire function and $0 < \lambda_p^{L^*}(f, h) < \infty$, $\sigma_p^{L^*}(g) < \infty$, where p is any positive integer. If h satisfy the Property (A) and $\exp^{[p-1]} L(M_g(r)) = o([r \exp L(r)]^\beta)$ as $r \rightarrow \infty$ and for some positive $\beta < \rho_p^{L^*}(g)$, then*

$$\lim_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(\exp[r \exp L(r)]^{\rho_p^{L^*}(g)})} \leq \sigma_p^{L^*}(g).$$

Proof. Let us consider that $\alpha > 2$ and $\delta \rightarrow 1^+$ in Lemma 14. Since $T_h^{-1}(r)$ is an increasing function of r , it follows from Lemma 9, Lemma 14 and the inequality $T_g(r) \leq \log M_g(r)$ ([13]) for a sequence of values of r tending to infinity that

$$\begin{aligned} T_h^{-1} T_{f \circ g}(r) &\leq T_h^{-1} [\{1 + o(1)\} T_f(M_g(r))], \\ \text{i.e. } T_h^{-1} T_{f \circ g}(r) &\leq \alpha [T_h^{-1} T_f(M_g(r))], \\ \text{i.e. } \log T_h^{-1} T_{f \circ g}(r) &\leq \log T_h^{-1} T_f(M_g(r)) + O(1), \\ \text{i.e. } \log T_h^{-1} T_{f \circ g}(r) &\leq (\lambda_p^{L^*}(f, h) + \varepsilon) [\log M_g(r) + \exp^{[p-1]} L(M_g(r))] + O(1), \\ \text{i.e. } \log T_h^{-1} T_{f \circ g}(r) &\leq (\lambda_p^{L^*}(f, h) + \varepsilon) \\ &\quad \times \left[(\sigma_p^{L^*}(g) + \varepsilon) [r \exp^{[p]} L(r)]^{\rho_p^{L^*}(g)} + \exp^{[p-1]} L(M_g(r)) \right] + O(1). \end{aligned} \tag{1}$$

Further in view of Lemma 3, we obtain for all sufficiently large values of r that

$$\begin{aligned} \log T_{P_0[h]}^{-1} T_{P_0[f]}(\exp[r \exp L(r)]^{\rho_p^{L^*}(g)}) \\ \geq (\lambda_p^{L^*}(P_0[f], P_0[h]) - \varepsilon) \left[[r \exp L(r)]^{\rho_p^{L^*}(g)} + \exp^{[p-1]} L(\exp[r \exp L(r)]^{\rho_p^{L^*}(g)}) \right], \end{aligned}$$

i.e. $\log T_{P_0[h]}^{-1} T_{P_0[f]} \left(\exp [r \exp L(r)]^{\rho_p^{L^*}(g)} \right) \geq \left(\lambda_p^{L^*}(f, h) - \varepsilon \right) \cdot [r \exp L(r)]^{\rho_p^{L^*}(g)}$.

Now from (1) and above it follows for a sequence of values of r tending to infinity that

$$\begin{aligned} & \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]} \left(\exp [r \exp L(r)]^{\rho_p^{L^*}(g)} \right)} \\ & \leq \frac{O(1) + \left(\lambda_p^{L^*}(f, h) + \varepsilon \right) \cdot \left[\left(\sigma_p^{L^*}(g) + \varepsilon \right) [r \exp L(r)]^{\rho_p^{L^*}(g)} + \exp^{[p-1] L(M_g(r))} \right]}{\left(\lambda_p^{L^*}(f, h) - \varepsilon \right) \cdot [r \exp L(r)]^{\rho_p^{L^*}(g)}}, \\ & \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]} \left(\exp [r \exp L(r)]^{\rho_p^{L^*}(g)} \right)} \leq \frac{O(1)}{\left(\lambda_p^{L^*}(f, h) - \varepsilon \right) \cdot [r \exp L(r)]^{\rho_p^{L^*}(g)}} \\ & + \frac{\left(\lambda_p^{L^*}(f, h) + \varepsilon \right) \cdot \left[\left(\sigma_p^{L^*}(g) + \varepsilon \right) + \frac{\exp^{[p-1] L(M_g(r))}}{[r \exp L(r)]^{\rho_p^{L^*}(g)}} \right]}{\left(\lambda_p^{L^*}(f, h) - \varepsilon \right)}. \end{aligned} \tag{2}$$

As $\beta < \rho_p^{L^*}(g)$ and $\exp^{[p-1] L(M_g(r))} = o\left([r \exp L(r)]^\beta\right)$ as $r \rightarrow \infty$, we obtain that

$$\lim_{r \rightarrow \infty} \frac{\exp^{[p-1] L(M_g(r))}}{[r \exp L(r)]^{\rho_p^{L^*}(g)}} = 0. \tag{3}$$

Since $\varepsilon (> 0)$ is arbitrary, it follows from (2) and (3) that

$$\varliminf_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]} \left(\exp [r \exp L(r)]^{\rho_p^{L^*}(g)} \right)} \leq \sigma_p^{L^*}(g).$$

Thus the theorem is established. \square

Remark 1. In Theorem 1 the condition $0 < \lambda_p^{L^*}(f, h) < \infty$ can be replaced by the condition $0 < \rho_p^{L^*}(f, h) < \infty$. If we will replace this condition by $0 < \lambda_p^{L^*}(f, h) \leq \rho_p^{L^*}(f, h) < \infty$, then

$$\overline{\lim}_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]} \left(\exp [r \exp L(r)]^{\rho_p^{L^*}(g)} \right)} \leq \frac{\rho_p^{L^*}(f, h) \cdot \sigma_p^{L^*}(g)}{\lambda_p^{L^*}(f, h)},$$

and if in addition we will replace the condition $\sigma_p^{L^*}(g) < \infty$ by $\overline{\sigma}_p^{L^*}(g) < \infty$ then

$$\overline{\lim}_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]} \left(\exp [r \exp L(r)]^{\rho_p^{L^*}(g)} \right)} \leq \frac{\rho_p^{L^*}(f, h) \cdot \overline{\sigma}_p^{L^*}(g)}{\lambda_p^{L^*}(f, h)}.$$

In the line of Theorem 1 and with the help of Lemma 4, one can easily prove the following theorem and therefore its proof is omitted.

Theorem 2. Let the meromorphic function f and entire function h satisfy the conditions of Lemma 4. Also let g be an entire function and $0 < \lambda_p^{L^*}(f, h) < \infty$, $\sigma_p^{L^*}(g) < \infty$, where p is any positive integer. If h satisfy the Property (A) and $\exp^{[p-1] L(M_g(r))} = o\left([r \exp L(r)]^\beta\right)$ as $r \rightarrow \infty$ and for some positive $\beta < \rho_p^{L^*}(g)$, then

$$\varliminf_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{M[h]}^{-1} T_{M[f]} \left(\exp [r \exp L(r)]^{\rho_p^{L^*}(g)} \right)} \leq \sigma_p^{L^*}(g).$$

Remark 2. In Theorem 2 the condition $0 < \lambda_p^{L^*}(f, h) < \infty$ can be replaced by the condition $0 < \rho_p^{L^*}(f, h) < \infty$. If we will replace this condition by $0 < \lambda_p^{L^*}(f, h) \leq \rho_p^{L^*}(f, h) < \infty$, then

$$\overline{\lim}_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{M[h]}^{-1} T_{M[f]} \left(\exp [r \exp L(r)]^{\rho_p^{L^*}(g)} \right)} \leq \frac{\rho_p^{L^*}(f, h) \cdot \sigma_p^{L^*}(g)}{\lambda_p^{L^*}(f, h)},$$

and if in addition we will replace the condition $\sigma_p^{L^*}(g) < \infty$ by $\bar{\sigma}_p^{L^*}(g) < \infty$ then

$$\lim_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{M[h]}^{-1} T_{M[f]} \left(\exp [r \exp L(r)]^{\rho_p^{L^*}(g)} \right)} \leq \frac{\rho_p^{L^*}(f, h) \cdot \bar{\sigma}_p^{L^*}(g)}{\lambda_p^{L^*}(f, h)}.$$

Now we state the following theorem without proof as it can be carried out in the line of Theorem 1.

Theorem 3. Let g be an entire function either of finite order or of non-zero lower order such that $\Theta(\infty; g) = \sum_{a \neq \infty} \delta_p(a; g) = 1$ or $\delta(\infty; g) = \sum_{a \neq \infty} \delta(a; g) = 1$ and k be an entire function with regular growth having non zero finite order and $\Theta(\infty; k) = \sum_{a \neq \infty} \delta_p(a; k) = 1$ or $\delta(\infty; k) = \sum_{a \neq \infty} \delta(a; k) = 1$. Also let f be a meromorphic function and h be an entire function such that $\lambda_p^{L^*}(f, h) < \infty$, $\lambda_p^{L^*}(g, k) > 0$ and $\sigma_p^{L^*}(g) < \infty$, where p is any positive integer. If h satisfy the Property (A) and $\exp^{[p-1]} L(M_g(r)) = o\left([r \exp L(r)]^\beta\right)$ as $r \rightarrow \infty$ and for some positive $\beta < \rho_p^{L^*}(g)$, then

$$\lim_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P_0[k]}^{-1} T_{P_0[g]} \left(\exp [r \exp L(r)]^{\rho_p^{L^*}(g)} \right)} \leq \frac{\lambda_p^{L^*}(f, h) \cdot \sigma_p^{L^*}(g)}{\lambda_p^{L^*}(g, k)}. \quad (4)$$

Remark 3. In Theorem 3, if we will replace the conditions $\lambda_p^{L^*}(f, h) < \infty$ and $\lambda_p^{L^*}(g, k) > 0$ by $\rho_p^{L^*}(f, h) < \infty$ and $\rho_p^{L^*}(g, k) > 0$ respectively, then is need to go the same replacement in right part of (4). Also if we will replace only the condition $\lambda_p^{L^*}(f, h) < \infty$ by $\rho_p^{L^*}(f, h) < \infty$ in Theorem 3, then

$$\overline{\lim}_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P_0[k]}^{-1} T_{P_0[g]} \left(\exp [r \exp L(r)]^{\rho_p^{L^*}(g)} \right)} \leq \frac{\rho_p^{L^*}(f, h) \cdot \sigma_p^{L^*}(g)}{\lambda_p^{L^*}(g, k)}.$$

Remark 4. In Theorem 3, if we will replace the conditions $\lambda_p^{L^*}(f, h) < \infty$ and $\sigma_p^{L^*}(g) < \infty$ by $\rho_p^{L^*}(f, h) < \infty$ and $\bar{\sigma}_p^{L^*}(g) < \infty$ respectively, then is need to go the same replacement in right part of (4).

In the line of Theorem 3 and with the help of Lemma 4, one can easily prove the following theorem and therefore its proof is omitted.

Theorem 4. Let g be a transcendental entire function of finite order or of non-zero lower order such that $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; g) = 4$ and k be a transcendental entire function with regular growth

and non zero finite order and $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; k) = 4$. Also let f be a meromorphic function and h be an entire function such that $\lambda_p^{L^*}(f, h) < \infty$, $\lambda_p^{L^*}(g, k) > 0$ and $\sigma_p^{L^*}(g) < \infty$, where p is any positive integer. If h satisfy the Property (A) and $\exp^{[p-1]L}(M_g(r)) = o\left([r \exp L(r)]^\beta\right)$ as $r \rightarrow \infty$ and for some positive $\beta < \rho_p^{L^*}(g)$, then

$$\overline{\lim}_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{M[k]}^{-1} T_{M[g]} \left(\exp [r \exp L(r)]^{\rho_p^{L^*}(g)} \right)} \leq \frac{\lambda_p^{L^*}(f, h) \cdot \sigma_p^{L^*}(g)}{\lambda_p^{L^*}(g, k)}. \tag{5}$$

Remark 5. In Theorem 4, if we will replace the conditions $\lambda_p^{L^*}(f, h) < \infty$ and $\lambda_p^{L^*}(g, k) > 0$ by $\rho_p^{L^*}(f, h) < \infty$ and $\rho_p^{L^*}(g, k) > 0$ respectively, then is need to go the same replacement in right part of (5). Also if we will replace only the condition $\lambda_p^{L^*}(f, h) < \infty$ by $\rho_p^{L^*}(f, h) < \infty$ in Theorem 4, then

$$\overline{\lim}_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{M[k]}^{-1} T_{M[g]} \left(\exp [r \exp L(r)]^{\rho_p^{L^*}(g)} \right)} \leq \frac{\rho_p^{L^*}(f, h) \cdot \sigma_p^{L^*}(g)}{\lambda_p^{L^*}(g, k)}.$$

Remark 6. In Theorem 4, if we will replace the conditions $\lambda_p^{L^*}(f, h) < \infty$ and $\sigma_p^{L^*}(g) < \infty$ by $\rho_p^{L^*}(f, h) < \infty$ and $\bar{\sigma}_p^{L^*}(g) < \infty$ respectively, then is need to go the same replacement in right part of (5).

Further we state the following two theorems which are based on pL^* -weak type.

Theorem 5. Let the meromorphic function f and entire function h satisfy the conditions of Lemma 3. Let g be an entire function and $0 < \lambda_p^{L^*}(f, h) \leq \rho_p^{L^*}(f, h) < \infty$, $\tau_p^{L^*}(g) < \infty$, where p is any positive integer. If h satisfy the Property (A) and $\exp^{[p-1]L}(M_g(r)) = o\left([r \exp L(r)]^\beta\right)$ as $r \rightarrow \infty$ and for some positive $\beta < \lambda_p^{L^*}(g)$, then

$$\overline{\lim}_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]} \left(\exp [r \exp L(r)]^{\lambda_p^{L^*}(g)} \right)} \leq \frac{\rho_p^{L^*}(f, h) \cdot \tau_p^{L^*}(g)}{\lambda_p^{L^*}(f, h)}.$$

Theorem 6. Let g be an entire function either of finite order or of non-zero lower order such that $\Theta(\infty; g) = \sum_{a \neq \infty} \delta_p(a; g) = 1$ or $\delta(\infty; g) = \sum_{a \neq \infty} \delta(a; g) = 1$ and k be an entire function with regular growth having non zero finite order and $\Theta(\infty; k) = \sum_{a \neq \infty} \delta_p(a; k) = 1$ or $\delta(\infty; k) = \sum_{a \neq \infty} \delta(a; k) = 1$. Also let f be a meromorphic function and h be an entire function such that $\rho_p^{L^*}(f, h) < \infty$, $\lambda_p^{L^*}(g, k) > 0$ and $\tau_p^{L^*}(g) < \infty$, where p is any positive integer. If h satisfy the Property (A) and $\exp^{[p-1]L}(M_g(r)) = o\left([r \exp L(r)]^\beta\right)$ as $r \rightarrow \infty$ and for some positive $\beta < \lambda_p^{L^*}(g)$, then

$$\overline{\lim}_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P_0[k]}^{-1} T_{P_0[g]} \left(\exp [r \exp L(r)]^{\lambda_p^{L^*}(g)} \right)} \leq \frac{\rho_p^{L^*}(f, h) \cdot \tau_p^{L^*}(g)}{\lambda_p^{L^*}(g, k)}.$$

The proofs of the above two theorems can be carried out in the line of Theorem 1 and Theorem 3 respectively and therefore their proofs are omitted.

In the line of Theorem 5 and Theorem 6 respectively and with the help of Lemma 4, one can easily prove the following two theorems and therefore their proofs are omitted.

Theorem 7. Let meromorphic function f and entire function h satisfy the conditions of Lemma 4. Also let g be an entire function and $0 < \lambda_p^{L^*}(f, h) \leq \rho_p^{L^*}(f, h) < \infty$, $\tau_p^{L^*}(g) < \infty$, where p is any positive integer. If h satisfy the Property (A) and $\exp^{[p-1]L}(M_g(r)) = o\left([r \exp L(r)]^\beta\right)$ as $r \rightarrow \infty$ and for some positive $\beta < \lambda_p^{L^*}(g)$, then

$$\lim_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{M[h]}^{-1} T_{M[f]} \left(\exp [r \exp L(r)]^{\lambda_p^{L^*}(g)} \right)} \leq \frac{\rho_p^{L^*}(f, h) \cdot \tau_p^{L^*}(g)}{\lambda_p^{L^*}(f, h)}.$$

Theorem 8. Let g be a transcendental entire function of finite order or of non-zero lower order such that $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; g) = 4$ and k be a transcendental entire function with regular growth and non zero finite order and $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; k) = 4$. Also let f be a meromorphic function and h be an entire function such that $\rho_p^{L^*}(f, h) < \infty$, $\lambda_p^{L^*}(g, k) > 0$ and $\tau_p^{L^*}(g) < \infty$, where p is any positive integer. If h satisfy the Property (A) and $\exp^{[p-1]L}(M_g(r)) = o\left([r \exp L(r)]^\beta\right)$ as $r \rightarrow \infty$ and for some positive $\beta < \lambda_p^{L^*}(g)$, then

$$\lim_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{M[k]}^{-1} T_{M[g]} \left(\exp [r \exp L(r)]^{\lambda_p^{L^*}(g)} \right)} \leq \frac{\rho_p^{L^*}(f, h) \cdot \tau_p^{L^*}(g)}{\lambda_p^{L^*}(g, k)}.$$

Using the concept of the growth indicator $\bar{\tau}_p^{L^*}(g)$ of an entire function g , we may state the subsequent two theorems without their proofs since those can be carried out in the line of Theorem 1 and Theorem 3 respectively.

Theorem 9. Let the meromorphic function f and entire function h satisfy the conditions of Lemma 3. Also let g be an entire function and $0 < \lambda_p^{L^*}(f, h) < \infty$, $\bar{\tau}_p^{L^*}(g) < \infty$, where p is any positive integer. If h satisfy the Property (A) and $\exp^{[p-1]L}(M_g(r)) = o\left([r \exp L(r)]^\beta\right)$ as $r \rightarrow \infty$ and for some positive $\beta < \lambda_p^{L^*}(g)$, then

$$\lim_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]} \left(\exp [r \exp L(r)]^{\lambda_p^{L^*}(g)} \right)} \leq \bar{\tau}_p^{L^*}(g).$$

Remark 7. In Theorem 9 the condition $0 < \lambda_p^{L^*}(f, h) < \infty$ can be replaced by the condition $0 < \rho_p^{L^*}(f, h) < \infty$. If we will replace this condition by $0 < \lambda_p^{L^*}(f, h) \leq \rho_p^{L^*}(f, h) < \infty$, then

$$\lim_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]} \left(\exp [r \exp L(r)]^{\lambda_p^{L^*}(g)} \right)} \leq \frac{\rho_p^{L^*}(f, h) \cdot \bar{\tau}_p^{L^*}(g)}{\lambda_p^{L^*}(f, h)}.$$

Theorem 10. Let entire functions g and k satisfy the conditions of Theorem 3. Let f be a meromorphic function and h be an entire function such that $\lambda_p^{L^*}(f, h) < \infty$, $\lambda_p^{L^*}(g, k) > 0$

and $\overline{\tau}_p^{L^*}(g) < \infty$, where p is any positive integer. If h satisfy the Property (A) and $\exp^{[p-1]} L(M_g(r)) = o\left([r \exp L(r)]^\beta\right)$ as $r \rightarrow \infty$ and for some positive $\beta < \lambda_p^{L^*}(g)$, then

$$\underline{\lim}_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P_0[k]}^{-1} T_{P_0[g]} \left(\exp [r \exp L(r)]^{\lambda_p^{L^*}(g)} \right)} \leq \frac{\lambda_p^{L^*}(f, h) \cdot \overline{\tau}_p^{L^*}(g)}{\lambda_p^{L^*}(g, k)}. \quad (6)$$

Remark 8. In Theorem 10, if we will replace the condition $\lambda_p^{L^*}(f, h) < \infty$ by $\rho_p^{L^*}(f, h) < \infty$, then is need to go the same replacement in right part of (6).

Remark 9. In Theorem 10, if we will replace the conditions $\lambda_p^{L^*}(f, h) < \infty$ and $\lambda_p^{L^*}(g, k) > 0$ by $\rho_p^{L^*}(f, h) < \infty$ and $\rho_p^{L^*}(g, k) > 0$ respectively, then is need to go the same replacement in right part of (6).

In the line of Theorem 9 and Theorem 10 respectively, one can easily prove the following six theorems and therefore their proofs are omitted.

Theorem 11. Let the meromorphic function f and entire function h satisfy the conditions of Lemma 4. Also let g be an entire function and $0 < \lambda_p^{L^*}(f, h) < \infty$, $\overline{\tau}_p^{L^*}(g) < \infty$, where p is any positive integer. If h satisfy the Property (A) and $\exp^{[p-1]} L(M_g(r)) = o\left([r \exp L(r)]^\beta\right)$ as $r \rightarrow \infty$ and for some positive $\beta < \lambda_p^{L^*}(g)$, then

$$\underline{\lim}_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{M[h]}^{-1} T_{M[f]} \left(\exp [r \exp L(r)]^{\lambda_p^{L^*}(g)} \right)} \leq \overline{\tau}_p^{L^*}(g).$$

Remark 10. In Theorem 11 the condition $0 < \lambda_p^{L^*}(f, h) < \infty$ can be replaced by the condition $0 < \rho_p^{L^*}(f, h) < \infty$. If we will replace this condition by $0 < \lambda_p^{L^*}(f, h) \leq \rho_p^{L^*}(f, h) < \infty$, then

$$\underline{\lim}_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{M[h]}^{-1} T_{M[f]} \left(\exp [r \exp L(r)]^{\lambda_p^{L^*}(g)} \right)} \leq \frac{\rho_p^{L^*}(f, h) \cdot \overline{\tau}_p^{L^*}(g)}{\lambda_p^{L^*}(f, h)}.$$

Theorem 12. Let the entire functions g and k satisfy the conditions of Theorem 4. Let f be a meromorphic function and h be an entire function such that $\lambda_p^{L^*}(f, h) < \infty$, $\lambda_p^{L^*}(g, k) > 0$ and $\overline{\tau}_p^{L^*}(g) < \infty$, where p is any positive integer. If h satisfy the Property (A) and $\exp^{[p-1]} L(M_g(r)) = o\left([r \exp L(r)]^\beta\right)$ as $r \rightarrow \infty$ and for some positive $\beta < \lambda_p^{L^*}(g)$, then

$$\underline{\lim}_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{M[k]}^{-1} T_{M[g]} \left(\exp [r \exp L(r)]^{\lambda_p^{L^*}(g)} \right)} \leq \frac{\lambda_p^{L^*}(f, h) \cdot \overline{\tau}_p^{L^*}(g)}{\lambda_p^{L^*}(g, k)}. \quad (7)$$

Remark 11. In Theorem 12, if we will replace the condition $\lambda_p^{L^*}(f, h) < \infty$ by $\rho_p^{L^*}(f, h) < \infty$, then is need to go the same replacement in right part of (7).

Remark 12. In Theorem 12, if we will replace the conditions $\lambda_p^{L^*}(f, h) < \infty$ and $\lambda_p^{L^*}(g, k) > 0$ by $\rho_p^{L^*}(f, h) < \infty$ and $\rho_p^{L^*}(g, k) > 0$ respectively, then is need to go the same replacement in right part of (7).

Theorem 13. Let the meromorphic function f and entire function h satisfy the conditions of Lemma 5. Also let g be an entire function and $0 < \rho_p^{L^*}(f, h) < \rho_g, \sigma_p^{L^*}(f, h) > 0$, where p is any positive integer. If $\exp^{[p-1]} L \left(\exp \left(re^{L(r)} \right)^\beta \right) = o \left(\left[r \exp^{[p]} L(r) \right]^\beta \right)$ ($r \rightarrow \infty$) for any $\beta > 0$, then

$$\lim_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g} \left(re^{L(r)} \right)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r)} \geq \frac{\lambda_p^{L^*}(f, h)}{\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}} \right)^{\frac{1}{\rho_h}} \cdot \sigma_p^{L^*}(f, h)}.$$

Proof. From the definition of relative pL^* -type of meromorphic function and in view of Lemma 5, we obtain for all sufficiently large values of r that

$$\begin{aligned} T_{P_0[h]}^{-1} T_{P_0[f]}(r) &\leq \left(\sigma_p^{L^*}(P_0[f], P_0[h]) + \varepsilon \right) \left[r \exp^{[p]} L(r) \right]^{\rho_p^{L^*}(P_0[f], P_0[h])}, \\ \text{i.e. } T_{P_0[h]}^{-1} T_{P_0[f]}(r) &\leq \left(\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}} \right)^{\frac{1}{\rho_h}} \cdot \sigma_p^{L^*}(f, h) + \varepsilon \right) \left[r \exp^{[p]} L(r) \right]^{\rho_p^{L^*}(f, h)}. \end{aligned} \quad (8)$$

As $0 < \rho_p^{L^*}(f, h) < \rho_g$, we obtain in view of Lemma 10 for a sequence of values of r tending to infinity that

$$\begin{aligned} \log T_h^{-1} T_{f \circ g} \left(re^{L(r)} \right) &\geq \log T_h^{-1} T_f \left(\exp \left(re^{L(r)} \right)^{\rho_p^{L^*}(f, h)} \right), \text{ i.e.} \\ \log T_h^{-1} T_{f \circ g} \left(re^{L(r)} \right) &\geq \left(\lambda_p^{L^*}(f, h) - \varepsilon \right) \left[\left[re^{L(r)} \right]^{\rho_p^{L^*}(f, h)} + \exp^{[p-1]} L \left(\exp \left(re^{L(r)} \right)^{\rho_p^{L^*}(f, h)} \right) \right]. \end{aligned}$$

Therefore from (8) and above, it follows for a sequence of values of r tending to infinity that

$$\frac{\log T_h^{-1} T_{f \circ g} \left(re^{L(r)} \right)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r)} \geq \frac{\left(\lambda_p^{L^*}(f, h) - \varepsilon \right) \left[\left[re^{L(r)} \right]^{\rho_p^{L^*}(f, h)} + \exp^{[p-1]} L \left(\exp \left(re^{L(r)} \right)^{\rho_p^{L^*}(f, h)} \right) \right]}{\left(\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}} \right)^{\frac{1}{\rho_h}} \cdot \sigma_p^{L^*}(f, h) + \varepsilon \right) \left[r \exp^{[p]} L(r) \right]^{\rho_p^{L^*}(f, h)}}.$$

Since $\lim_{r \rightarrow \infty} \frac{\exp^{[p-1]} L \left(\exp \left(re^{L(r)} \right)^{\rho_p^{L^*}(f, h)} \right)}{\left[r \exp^{[p]} L(r) \right]^{\rho_p^{L^*}(f, h)}} = 0$ as $\exp^{[p-1]} L \left(\exp \left(re^{L(r)} \right)^\beta \right) = o \left(\left[r \exp^{[p]} L(r) \right]^\beta \right)$ ($r \rightarrow \infty$) for any $\alpha > 0$, we obtain from above that

$$\lim_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g} \left(re^{L(r)} \right)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r)} \geq \frac{\lambda_p^{L^*}(f, h)}{\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}} \right)^{\frac{1}{\rho_h}} \cdot \sigma_p^{L^*}(f, h)}.$$

Thus the theorem follows. \square

Remark 13. If we take $\bar{\tau}_p^{L^*}(f, h) > 0$ instead of $\sigma_p^{L^*}(f, h) > 0$ and the other conditions remain the same, then with the help of Lemma 6, one can easily verify that the conclusion of Theorem 13 remains valid with $\sigma_p^{L^*}(f, h)$ replaced by $\bar{\tau}_p^{L^*}(f, h)$.

In the line of Theorem 13 and in view of Lemma 7, one can easily prove the following theorem and therefore its proofs is omitted.

Theorem 14. *Let the meromorphic function f and entire function h satisfy the conditions of Lemma 7. Also let g be an entire function and $0 < \rho_p^{L^*}(f, h) < \rho_g, \sigma_p^{L^*}(f, h) > 0$, where p is any positive integer. If $\exp^{[p-1]} L \left(\exp \left(re^{L(r)} \right)^\beta \right) = o \left(\left[r \exp^{[p]} L(r) \right]^\beta \right)$ ($r \rightarrow \infty$) for any $\beta > 0$, then*

$$\overline{\lim}_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g} \left(re^{L(r)} \right)}{T_{M[h]}^{-1} T_{M[f]}(r)} \geq \frac{\lambda_p^{L^*}(f, h)}{\left(\frac{\Gamma_{M[f]} - (\Gamma_{M[f]} - \gamma_{M[f]}) \Theta(\infty; f)}{\Gamma_{M[h]} - (\Gamma_{M[h]} - \gamma_{M[h]}) \Theta(\infty; h)} \right)^{\frac{1}{p_h}} \cdot \sigma_p^{L^*}(f, h)}.$$

Remark 14. *If we take $\overline{\tau}_p^{L^*}(f, h) > 0$ instead of $\sigma_p^{L^*}(f, h) > 0$ and the other conditions remain the same, then with the help of Lemma 8, one can easily verify that the conclusion of Theorem 14 remains valid with $\sigma_p^{L^*}(f, h)$ replaced by $\overline{\tau}_p^{L^*}(f, h)$.*

Theorem 15. *Let the meromorphic function f and entire function h satisfy the conditions of Lemma 5. Also let g be an entire function, h satisfy the Property (A), $\rho_p^{L^*}(f, h) = \rho_p^{L^*}(g)$, $\sigma_p^{L^*}(g) < \infty$ and $\overline{\sigma}_p^{L^*}(f, h) > 0$, where p is any positive integer.*

(a) *If $\exp^{[p-1]} L(M_g(r)) = o \left\{ T_{P_0[h]}^{-1} T_{P_0[f]}(r) \right\}$ then*

$$\overline{\lim}_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r) + \exp^{[p-1]} L(M_g(r))} \leq \frac{\rho_p^{L^*}(f, h) \cdot \sigma_p^{L^*}(g)}{\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}} \right)^{\frac{1}{p_h}} \cdot \overline{\sigma}_p^{L^*}(f, h)}.$$

(b) *If $T_{P_0[h]}^{-1} T_{P_0[f]}(r) = o \left\{ \exp^{[p-1]} L(M_g(r)) \right\}$ then*

$$\overline{\lim}_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r) + \exp^{[p-1]} L(M_g(r))} \leq \rho_p^{L^*}(f, h).$$

Proof. Let us consider that $\alpha > 2$ and $\delta \rightarrow 1^+$ in Lemma 14. Since $T_h^{-1}(r)$ is an increasing function of r , it follows from Lemma 9, Lemma 14 and the inequality $T_g(r) \leq \log M_g(r)$ (cf. [13]) for all sufficiently large values of r that

$$\begin{aligned} T_h^{-1} T_{f \circ g}(r) &\leq T_h^{-1} \left[\{1 + o(1)\} T_f(M_g(r)) \right], \\ \text{i.e. } T_h^{-1} T_{f \circ g}(r) &\leq \alpha \left[T_h^{-1} T_f(M_g(r)) \right], \\ \text{i.e. } \log T_h^{-1} T_{f \circ g}(r) &\leq \log T_h^{-1} T_f(M_g(r)) + O(1), \\ \text{i.e. } \log T_h^{-1} T_{f \circ g}(r) &\leq \left(\rho_p^{L^*}(f, h) + \varepsilon \right) \left(\log M_g(r) + \exp^{[p-1]} L(M_g(r)) \right) + O(1), \\ \text{i.e. } \log T_h^{-1} T_{f \circ g}(r) &\leq \left(\rho_p^{L^*}(f, h) + \varepsilon \right) \left(\sigma_p^{L^*}(g) + \varepsilon \right) \left[r \exp^{[p]} L(r) \right]^{\rho_p^{L^*}(g)} \\ &\quad + \left(\rho_p^{L^*}(f, h) + \varepsilon \right) \exp^{[p-1]} L(M_g(r)) + O(1). \end{aligned}$$

In view of condition (ii) we obtain from above for all sufficiently large values of r that

$$\begin{aligned} \log T_h^{-1} T_{f \circ g}(r) &\leq \left(\rho_p^{L^*}(f, h) + \varepsilon \right) \left(\sigma_p^{L^*}(g) + \varepsilon \right) \left[r \exp^{[p]} L(r) \right]^{\rho_p^{L^*}(f, h)} \\ &\quad + \left(\rho_p^{L^*}(f, h) + \varepsilon \right) \exp^{[p-1]} L(M_g(r)) + O(1). \end{aligned} \tag{9}$$

Again from the definition of relative ${}_pL^*$ -lower type we get in view of Lemma 5, for all sufficiently large values of r that

$$\begin{aligned} T_{P_0[h]}^{-1} T_{P_0[f]}(r) &\geq \left(\bar{\sigma}_p^{L^*}(P_0[f], P_0[h]) - \varepsilon \right) \left[r \exp^{[p]} L(r) \right]^{\rho_p^{L^*}(P_0[f], P_0[h])}, \text{ i.e.} \\ T_{P_0[h]}^{-1} T_{P_0[f]}(r) &\geq \left(\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}} \right)^{\frac{1}{\rho_h}} \cdot \bar{\sigma}_p^{L^*}(f, h) - \varepsilon \right) \left[r \exp^{[p]} L(r) \right]^{\rho_p^{L^*}(f, h)}, \text{ i.e.} \\ \left[r \exp^{[p]} L(r) \right]^{\rho_p^{L^*}(f, h)} &\leq \frac{T_{P_0[h]}^{-1} T_{P_0[f]}(r)}{\left(\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}} \right)^{\frac{1}{\rho_h}} \cdot \bar{\sigma}_p^{L^*}(f, h) - \varepsilon \right)}. \end{aligned} \quad (10)$$

Now from (9) and (10), it follows for all sufficiently large values of r that

$$\begin{aligned} \log T_h^{-1} T_{f \circ g}(r) &\leq \left(\rho_p^{L^*}(f, h) + \varepsilon \right) \left(\sigma_p^{L^*}(g) + \varepsilon \right) \frac{T_{P_0[h]}^{-1} T_{P_0[f]}(r)}{\left(\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}} \right)^{\frac{1}{\rho_h}} \cdot \bar{\sigma}_p^{L^*}(f, h) - \varepsilon \right)} \\ &\quad + \left(\rho_p^{L^*}(f, h) + \varepsilon \right) \exp^{[p-1]} L(M_g(r)) + O(1), \\ \text{i.e. } \frac{\log T_h^{-1} T_{f \circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r) + \exp^{[p-1]} L(M_g(r))} &\leq \frac{O(1)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r) + \exp^{[p-1]} L(M_g(r))} \\ &\quad + \frac{\left(\rho_p^{L^*}(f, h) + \varepsilon \right) \left(\sigma_p^{L^*}(g) + \varepsilon \right)}{\left(\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}} \right)^{\frac{1}{\rho_h}} \cdot \bar{\sigma}_p^{L^*}(f, h) - \varepsilon \right)} \\ &\quad + \frac{\left(\rho_p^{L^*}(f, h) + \varepsilon \right)}{1 + \frac{\exp^{[p-1]} L(M_g(r))}{T_{P_0[h]}^{-1} T_{P_0[f]}(r)}} + \frac{\left(\rho_p^{L^*}(f, h) + \varepsilon \right)}{1 + \frac{T_{P_0[h]}^{-1} T_{P_0[f]}(r)}{\exp^{[p-1]} L(M_g(r))}}. \end{aligned} \quad (11)$$

If $\exp^{[p-1]} L(M_g(r)) = o \left\{ T_{P_0[h]}^{-1} T_{P_0[f]}(r) \right\}$ then from (11) we get that

$$\overline{\lim}_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r) + \exp^{[p-1]} L(M_g(r))} \leq \frac{\left(\rho_p^{L^*}(f, h) + \varepsilon \right) \left(\sigma_p^{L^*}(g) + \varepsilon \right)}{\left(\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}} \right)^{\frac{1}{\rho_h}} \cdot \bar{\sigma}_p^{L^*}(f, h) - \varepsilon \right)}.$$

Since $\varepsilon (> 0)$ is arbitrary, it follows from above that

$$\overline{\lim}_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r) + \exp^{[p-1]} L(M_g(r))} \leq \frac{\rho_p^{L^*}(f, h) \cdot \sigma_p^{L^*}(g)}{\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}} \right)^{\frac{1}{\rho_h}} \cdot \bar{\sigma}_p^{L^*}(f, h)}.$$

Thus the first part of the theorem follows.

Since $\varepsilon (> 0)$ is arbitrary, and if $T_{P_0[h]}^{-1} T_{P_0[f]}(r) = o \left\{ \exp^{[p-1]} L(M_g(r)) \right\}$ then from (11) it follows that

$$\overline{\lim}_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r) + \exp^{[p-1]} L(M_g(r))} \leq \rho_p^{L^*}(f, h).$$

Thus the second part of the theorem is established. \square

Theorem 16. *Let the meromorphic function f and entire function h satisfy the conditions of Lemma 5. Also let g be an entire function, h satisfy the Property (A), $\rho_p^{L^*}(f, h) = \rho_p^{L^*}(g)$, $\lambda_p^{L^*}(f, h) < \infty$, $\bar{\sigma}_p^{L^*}(f, h) > 0$ and $\sigma_p^{L^*}(g) < \infty$, where p is any positive integer.*

(a) *If $\exp^{[p-1]} L(M_g(r)) = o\{T_{P_0[h]}^{-1} T_{P_0[f]}(r)\}$ then*

$$\overline{\lim}_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r) + \exp^{[p-1]} L(M_g(r))} \leq \frac{\lambda_p^{L^*}(f, h) \cdot \sigma_p^{L^*}(g)}{\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}} \cdot \bar{\sigma}_p^{L^*}(f, h)}.$$

(b) *If $T_{P_0[h]}^{-1} T_{P_0[f]}(r) = o\{\exp^{[p-1]} L(M_g(r))\}$ then*

$$\overline{\lim}_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r) + \exp^{[p-1]} L(M_g(r))} \leq \lambda_p^{L^*}(f, h).$$

We omit the proof of the above theorem as it can be carried out in the line of Theorem 15.

Remark 15. *In Theorem 16, if we take $\rho_p^{L^*}(f, h) = \rho_p^{L^*}(g)$, $\sigma_p^{L^*}(f, h) > 0$ and $\sigma_p^{L^*}(g) < \infty$ instead of $\rho_p^{L^*}(f, h) = \rho_p^{L^*}(g)$, $\lambda_p^{L^*}(f, h) < \infty$, $\bar{\sigma}_p^{L^*}(f, h) > 0$ and $\sigma_p^{L^*}(g) < \infty$ and the other conditions remain the same, then one can easily verify that the conclusion of Theorem 16 remains valid with $\lambda_p^{L^*}(f, h)$ replaced by $\rho_p^{L^*}(f, h)$ and $\bar{\sigma}_p^{L^*}(f, h)$ replaced by $\sigma_p^{L^*}(f, h)$ respectively.*

Remark 16. *In Theorem 16, if we take $\rho_p^{L^*}(f, h) = \rho_p^{L^*}(g)$, $\bar{\sigma}_p^{L^*}(f, h) > 0$ and $\bar{\sigma}_p^{L^*}(g) < \infty$ instead of $\rho_p^{L^*}(f, h) = \rho_p^{L^*}(g)$, $\lambda_p^{L^*}(f, h) < \infty$, $\bar{\sigma}_p^{L^*}(f, h) > 0$ and $\sigma_p^{L^*}(g) < \infty$ and the other conditions remain the same, then one can easily verify that the conclusion of Theorem 16 remains valid with $\lambda_p^{L^*}(f, h)$ replaced by $\rho_p^{L^*}(f, h)$ and $\sigma_p^{L^*}(g)$ replaced by $\bar{\sigma}_p^{L^*}(g)$ respectively.*

Similarly using the concept of the growth indicator $\tau_p^{L^*}(f, h)$ and $\bar{\tau}_p^{L^*}(g)$ we may state the subsequent two theorems without their proofs since those can be carried out in view of Lemma 6 and in the line of Theorem 15 and Theorem 16 respectively.

Theorem 17. *Let the meromorphic function f and entire function h satisfy the conditions of Lemma 6. Also let g be an entire function, h satisfy the Property (A), $\rho_p^{L^*}(f, h) < \infty$, $\lambda_p^{L^*}(f, h) = \lambda_p^{L^*}(g)$, $\bar{\tau}_p^{L^*}(g) < \infty$ and $\tau_p^{L^*}(f, h) > 0$, where p is any positive integer.*

(a) *If $\exp^{[p-1]} L(M_g(r)) = o\{T_{P_0[h]}^{-1} T_{P_0[f]}(r)\}$ then*

$$\overline{\lim}_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r) + \exp^{[p-1]} L(M_g(r))} \leq \frac{\rho_p^{L^*}(f, h) \cdot \bar{\tau}_p^{L^*}(g)}{\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}} \cdot \tau_p^{L^*}(f, h)}.$$

(b) *If $T_{P_0[h]}^{-1} T_{P_0[f]}(r) = o\{\exp^{[p-1]} L(M_g(r))\}$ then*

$$\overline{\lim}_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r) + \exp^{[p-1]} L(M_g(r))} \leq \rho_p^{L^*}(f, h).$$

Remark 17. *In Theorem 17, if we replace the condition $\lambda_p^{L^*}(f, h) = \lambda_p^{L^*}(g)$ and $\bar{\tau}_p^{L^*}(g) < \infty$ by $\lambda_p^{L^*}(f, h) = \rho_p^{L^*}(g)$ and $\sigma_p^{L^*}(g) < \infty$ and the other conditions remain the same, then one can easily verify that the conclusion of Theorem 17 remains valid with $\bar{\tau}_p^{L^*}(g)$ replaced by $\sigma_p^{L^*}(g)$.*

Remark 18. In Theorem 17, if we take $\rho_p^{L^*}(f, h) = \lambda_p^{L^*}(g)$, $\bar{\tau}_p^{L^*}(g) < \infty$ and $\bar{\sigma}_p^{L^*}(f, h) > 0$ instead of $\rho_p^{L^*}(f, h) < \infty$, $\lambda_p^{L^*}(f, h) = \lambda_p^{L^*}(g)$, $\bar{\tau}_p^{L^*}(g) < \infty$ and $\tau_p^{L^*}(f, h) > 0$ and the other conditions remain the same, then one can easily verify that the conclusion of Theorem 17 remains valid with $\tau_p^{L^*}(f, h)$ replaced by $\bar{\sigma}_p^{L^*}(f, h)$.

Theorem 18. Let the meromorphic function f and entire function h satisfy the conditions of Lemma 6. Also let g be an entire function, h satisfy the Property (A), $\lambda_p^{L^*}(f, h) = \lambda_p^{L^*}(g)$, $\bar{\tau}_p^{L^*}(g) < \infty$ and $\tau_p^{L^*}(f, h) > 0$, where p is any positive integer.

(a) If $\exp^{[p-1]} L(M_g(r)) = o\left\{T_{P_0[h]}^{-1} T_{P_0[f]}(r)\right\}$ then

$$\lim_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r) + \exp^{[p-1]} L(M_g(r))} \leq \frac{\lambda_p^{L^*}(f, h) \cdot \bar{\tau}_p^{L^*}(g)}{\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{p_h}} \cdot \tau_p^{L^*}(f, h)}.$$

(b) If $T_{P_0[h]}^{-1} T_{P_0[f]}(r) = o\left\{\exp^{[p-1]} L(M_g(r))\right\}$ then

$$\lim_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r) + \exp^{[p-1]} L(M_g(r))} \leq \lambda_p^{L^*}(f, h).$$

Remark 19. In Theorem 18, if we take $\rho_p^{L^*}(f, h) < \infty$, $\lambda_p^{L^*}(f, h) = \lambda_p^{L^*}(g)$, $\bar{\tau}_p^{L^*}(g) < \infty$ and $\bar{\tau}_p^{L^*}(f, h) > 0$ instead of $\lambda_p^{L^*}(f, h) = \lambda_p^{L^*}(g)$, $\bar{\tau}_p^{L^*}(g) < \infty$ and $\tau_p^{L^*}(f, h) > 0$ and the other conditions remain the same, then one can easily verify that the conclusion of Theorem 18 remains valid with $\lambda_p^{L^*}(f, h)$ replaced by $\rho_p^{L^*}(f, h)$ and $\tau_p^{L^*}(f, h)$ replaced by $\bar{\tau}_p^{L^*}(f, h)$ respectively.

Remark 20. In Theorem 18, if we take $\rho_p^{L^*}(f, h) < \infty$, $\lambda_p^{L^*}(f, h) = \lambda_p^{L^*}(g)$, $\tau_p^{L^*}(g) < \infty$ and $\tau_p^{L^*}(f, h) > 0$ instead of $\lambda_p^{L^*}(f, h) = \lambda_p^{L^*}(g)$, $\bar{\tau}_p^{L^*}(g) < \infty$ and $\tau_p^{L^*}(f, h) > 0$ and the other conditions remain the same, then one can easily verify that the conclusion of Theorem 18 remains valid with $\lambda_p^{L^*}(f, h)$ replaced by $\rho_p^{L^*}(f, h)$ and $\bar{\tau}_p^{L^*}(g)$ replaced by $\tau_p^{L^*}(g)$ respectively.

Remark 21. In Theorem 18, if we replace the conditions $\lambda_p^{L^*}(f, h) = \lambda_p^{L^*}(g)$ and $\bar{\tau}_p^{L^*}(g) < \infty$ by $\lambda_p^{L^*}(f, h) = \rho_p^{L^*}(g)$ and $\sigma_p^{L^*}(g) < \infty$ and the other conditions remain the same, then one can easily verify that the conclusion of Theorem 18 remains valid with $\bar{\tau}_p^{L^*}(g)$ replaced by $\sigma_p^{L^*}(g)$.

Remark 22. In Theorem 18, if we take $\rho_p^{L^*}(f, h) = \lambda_p^{L^*}(g)$, $\bar{\tau}_p^{L^*}(g) < \infty$ and $\bar{\sigma}_p^{L^*}(f, h) > 0$ instead of $\lambda_p^{L^*}(f, h) = \lambda_p^{L^*}(g)$, $\bar{\tau}_p^{L^*}(g) < \infty$ and $\tau_p^{L^*}(f, h) > 0$ and the other conditions remain the same, then one can easily verify that the conclusion of Theorem 18 remains valid with $\tau_p^{L^*}(f, h)$ replaced by $\bar{\sigma}_p^{L^*}(f, h)$.

In the line of Theorem 15, Theorem 16, Theorem 17 and Theorem 18 and in view of Lemma 7 and Lemma 8, one can easily prove the following four theorems and therefore their proofs are omitted.

Theorem 19. Let the meromorphic function f and entire function h satisfy the conditions of Lemma 7. Also let g be an entire function, h satisfy the Property (A), $\rho_p^{L^*}(f, h) = \rho_p^{L^*}(g)$,

$\sigma_p^{L^*}(g) < \infty$ and $\bar{\sigma}_p^{L^*}(f, h) > 0$, where p is any positive integer.

(a) If $\exp^{[p-1]L}(M_g(r)) = o\left\{T_{P_0[h]}^{-1}T_{P_0[f]}(r)\right\}$ then

$$\overline{\lim}_{r \rightarrow \infty} \frac{\log T_h^{-1}T_{f \circ g}(r)}{T_{M[h]}^{-1}T_{M[f]}(r) + \exp^{[p-1]L}(M_g(r))} \leq \frac{\rho_p^{L^*}(f, h) \cdot \sigma_p^{L^*}(g)}{\left(\frac{\Gamma_{M[f]} - (\Gamma_{M[f]} - \gamma_{M[f]})\Theta(\infty; f)}{\Gamma_{M[h]} - (\Gamma_{M[h]} - \gamma_{M[h]})\Theta(\infty; h)}\right)^{\frac{1}{\rho_h}} \cdot \bar{\sigma}_p^{L^*}(f, h)}.$$

(b) If $T_{P_0[h]}^{-1}T_{P_0[f]}(r) = o\left\{\exp^{[p-1]L}(M_g(r))\right\}$ then

$$\overline{\lim}_{r \rightarrow \infty} \frac{\log T_h^{-1}T_{f \circ g}(r)}{T_{M[h]}^{-1}T_{M[f]}(r) + \exp^{[p-1]L}(M_g(r))} \leq \rho_p^{L^*}(f, h).$$

Theorem 20. Let the meromorphic function f and entire function h satisfy the conditions of Lemma 7. Also let g be an entire function, h satisfy the Property (A), $\lambda_p^{L^*}(f, h) < \infty$, $\rho_p^{L^*}(f, h) = \rho_p^{L^*}(g)$, $\sigma_p^{L^*}(g) < \infty$ and $\bar{\sigma}_p^{L^*}(f, h) > 0$, where p is any positive integer.

(a) If $\exp^{[p-1]L}(M_g(r)) = o\left\{T_{P_0[h]}^{-1}T_{P_0[f]}(r)\right\}$ then

$$\overline{\lim}_{r \rightarrow \infty} \frac{\log T_h^{-1}T_{f \circ g}(r)}{T_{M[h]}^{-1}T_{M[f]}(r) + \exp^{[p-1]L}(M_g(r))} \leq \frac{\lambda_p^{L^*}(f, h) \cdot \sigma_p^{L^*}(g)}{\left(\frac{\Gamma_{M[f]} - (\Gamma_{M[f]} - \gamma_{M[f]})\Theta(\infty; f)}{\Gamma_{M[h]} - (\Gamma_{M[h]} - \gamma_{M[h]})\Theta(\infty; h)}\right)^{\frac{1}{\rho_h}} \cdot \bar{\sigma}_p^{L^*}(f, h)}.$$

(b) If $T_{P_0[h]}^{-1}T_{P_0[f]}(r) = o\left\{\exp^{[p-1]L}(M_g(r))\right\}$ then

$$\overline{\lim}_{r \rightarrow \infty} \frac{\log T_h^{-1}T_{f \circ g}(r)}{T_{M[h]}^{-1}T_{M[f]}(r) + \exp^{[p-1]L}(M_g(r))} \leq \lambda_p^{L^*}(f, h).$$

Remark 23. In Theorem 20, if we take $\rho_p^{L^*}(f, h) = \rho_p^{L^*}(g)$, $\sigma_p^{L^*}(f, h) > 0$ and $\sigma_p^{L^*}(g) < \infty$ instead of $\rho_p^{L^*}(f, h) = \rho_p^{L^*}(g)$, $\lambda_p^{L^*}(f, h) < \infty$, $\bar{\sigma}_p^{L^*}(f, h) > 0$ and $\sigma_p^{L^*}(g) < \infty$ and the other conditions remain the same, then one can easily verify that the conclusion of Theorem 20 remains valid with $\lambda_p^{L^*}(f, h)$ replaced by $\rho_p^{L^*}(f, h)$ and $\bar{\sigma}_p^{L^*}(f, h)$ replaced by $\sigma_p^{L^*}(f, h)$ respectively.

Remark 24. In Theorem 20, if we take $\rho_p^{L^*}(f, h) = \rho_p^{L^*}(g)$, $\bar{\sigma}_p^{L^*}(f, h) > 0$ and $\bar{\sigma}_p^{L^*}(g) < \infty$ instead of $\rho_p^{L^*}(f, h) = \rho_p^{L^*}(g)$, $\lambda_p^{L^*}(f, h) < \infty$, $\bar{\sigma}_p^{L^*}(f, h) > 0$ and $\sigma_p^{L^*}(g) < \infty$ and the other conditions remain the same, then one can easily verify that the conclusion of Theorem 20 remains valid with $\lambda_p^{L^*}(f, h)$ replaced by $\rho_p^{L^*}(f, h)$ and $\sigma_p^{L^*}(g)$ replaced by $\bar{\sigma}_p^{L^*}(g)$ respectively.

Theorem 21. Let the meromorphic function f and entire function h satisfy the conditions of Lemma 8. Also let g be an entire function, h satisfy the Property (A), $\rho_p^{L^*}(f, h) < \infty$, $\lambda_p^{L^*}(f, h) = \lambda_p^{L^*}(g)$, $\bar{\tau}_p^{L^*}(g) < \infty$ and $\tau_p^{L^*}(f, h) > 0$, where p is any positive integer.

(a) If $\exp^{[p-1]L}(M_g(r)) = o\left\{T_{P_0[h]}^{-1}T_{P_0[f]}(r)\right\}$ then

$$\overline{\lim}_{r \rightarrow \infty} \frac{\log T_h^{-1}T_{f \circ g}(r)}{T_{M[h]}^{-1}T_{M[f]}(r) + \exp^{[p-1]L}(M_g(r))} \leq \frac{\rho_p^{L^*}(f, h) \cdot \bar{\tau}_p^{L^*}(g)}{\left(\frac{\Gamma_{M[f]} - (\Gamma_{M[f]} - \gamma_{M[f]})\Theta(\infty; f)}{\Gamma_{M[h]} - (\Gamma_{M[h]} - \gamma_{M[h]})\Theta(\infty; h)}\right)^{\frac{1}{\rho_h}} \cdot \tau_p^{L^*}(f, h)}.$$

(b) If $T_{P_0[h]}^{-1} T_{P_0[f]}(r) = o \left\{ \exp^{[p-1]} L(M_g(r)) \right\}$ then

$$\overline{\lim}_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{T_{M[h]}^{-1} T_{M[f]}(r) + \exp^{[p-1]} L(M_g(r))} \leq \rho_p^{L^*}(f, h).$$

Remark 25. In Theorem 21, if we replace the condition $\lambda_p^{L^*}(f, h) = \lambda_p^{L^*}(g)$ and $\overline{\tau}_p^{L^*}(g) < \infty$ by $\lambda_p^{L^*}(f, h) = \rho_p^{L^*}(g)$ and $\sigma_p^{L^*}(g) < \infty$ and the other conditions remain the same, then one can easily verify that the conclusion of Theorem 21 remains valid with $\overline{\tau}_p^{L^*}(g)$ replaced by $\sigma_p^{L^*}(g)$.

Remark 26. In Theorem 21, if we take $\rho_p^{L^*}(f, h) = \lambda_p^{L^*}(g)$, $\overline{\tau}_p^{L^*}(g) < \infty$ and $\overline{\sigma}_p^{L^*}(f, h) > 0$ instead of $\rho_p^{L^*}(f, h) < \infty$, $\lambda_p^{L^*}(f, h) = \lambda_p^{L^*}(g)$, $\overline{\tau}_p^{L^*}(g) < \infty$ and $\tau_p^{L^*}(f, h) > 0$ and the other conditions remain the same, then one can easily verify that the conclusion of Theorem 21 remains valid with $\tau_p^{L^*}(f, h)$ replaced by $\overline{\sigma}_p^{L^*}(f, h)$.

Theorem 22. Let the meromorphic function f and entire function h satisfy the conditions of Lemma 8. Also let g be an entire function, h satisfy the Property (A), $\lambda_p^{L^*}(f, h) = \lambda_p^{L^*}(g)$, $\overline{\tau}_p^{L^*}(g) < \infty$ and $\tau_p^{L^*}(f, h) > 0$, where p is any positive integer.

(a) If $\exp^{[p-1]} L(M_g(r)) = o \left\{ T_{P_0[h]}^{-1} T_{P_0[f]}(r) \right\}$ then

$$\overline{\lim}_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{T_{M[h]}^{-1} T_{M[f]}(r) + \exp^{[p-1]} L(M_g(r))} \leq \frac{\lambda_p^{L^*}(f, h) \cdot \overline{\tau}_p^{L^*}(g)}{\left(\frac{\Gamma_{M[f]} - (\Gamma_{M[f]} - \gamma_{M[f]}) \Theta(\infty; f)}{\Gamma_{M[h]} - (\Gamma_{M[h]} - \gamma_{M[h]}) \Theta(\infty; h)} \right)^{\frac{1}{\rho_h}} \cdot \tau_p^{L^*}(f, h)}.$$

(b) If $T_{P_0[h]}^{-1} T_{P_0[f]}(r) = o \left\{ \exp^{[p-1]} L(M_g(r)) \right\}$ then

$$\overline{\lim}_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{T_{M[h]}^{-1} T_{M[f]}(r) + \exp^{[p-1]} L(M_g(r))} \leq \lambda_p^{L^*}(f, h).$$

Remark 27. In Theorem 22, if we take $\rho_p^{L^*}(f, h) < \infty$, $\lambda_p^{L^*}(f, h) = \lambda_p^{L^*}(g)$, $\overline{\tau}_p^{L^*}(g) < \infty$ and $\overline{\tau}_p^{L^*}(f, h) > 0$ instead of $\lambda_p^{L^*}(f, h) = \lambda_p^{L^*}(g)$, $\overline{\tau}_p^{L^*}(g) < \infty$ and $\tau_p^{L^*}(f, h) > 0$ and the other conditions remain the same, then one can easily verify that the conclusion of Theorem 22 remains valid with $\lambda_p^{L^*}(f, h)$ replaced by $\rho_p^{L^*}(f, h)$ and $\tau_p^{L^*}(f, h)$ replaced by $\overline{\tau}_p^{L^*}(f, h)$ respectively.

Remark 28. In Theorem 22, if we take $\rho_p^{L^*}(f, h) < \infty$, $\lambda_p^{L^*}(f, h) = \lambda_p^{L^*}(g)$, $\tau_p^{L^*}(g) < \infty$ and $\tau_p^{L^*}(f, h) > 0$ instead of $\lambda_p^{L^*}(f, h) = \lambda_p^{L^*}(g)$, $\overline{\tau}_p^{L^*}(g) < \infty$ and $\tau_p^{L^*}(f, h) > 0$ and the other conditions remain the same, then one can easily verify that the conclusion of Theorem 22 remains valid with $\lambda_p^{L^*}(f, h)$ replaced by $\rho_p^{L^*}(f, h)$ and $\overline{\tau}_p^{L^*}(g)$ replaced by $\tau_p^{L^*}(g)$ respectively.

Remark 29. In Theorem 22, if we replace the condition $\lambda_p^{L^*}(f, h) = \lambda_p^{L^*}(g)$ and $\overline{\tau}_p^{L^*}(g) < \infty$ by $\lambda_p^{L^*}(f, h) = \rho_p^{L^*}(g)$ and $\sigma_p^{L^*}(g) < \infty$ and the other conditions remain the same, then one can easily verify that the conclusion of Theorem 22 remains valid with $\overline{\tau}_p^{L^*}(g)$ replaced by $\sigma_p^{L^*}(g)$.

Remark 30. In Theorem 22, if we take $\rho_p^{L^*}(f, h) = \lambda_p^{L^*}(g)$, $\overline{\tau}_p^{L^*}(g) < \infty$ and $\overline{\sigma}_p^{L^*}(f, h) > 0$ instead of $\lambda_p^{L^*}(f, h) = \lambda_p^{L^*}(g)$, $\overline{\tau}_p^{L^*}(g) < \infty$ and $\tau_p^{L^*}(f, h) > 0$ and the other conditions remain the same, then one can easily verify that the conclusion of Theorem 22 remains valid with $\tau_p^{L^*}(f, h)$ replaced by $\overline{\sigma}_p^{L^*}(f, h)$.

Theorem 23. Let f be a meromorphic function either of finite order or of non-zero lower order such that $\Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1$ and h be an entire function having regular growth and non zero finite order with $\Theta(\infty; h) = \sum_{a \neq \infty} \delta_p(a; h) = 1$ or $\delta(\infty; h) = \sum_{a \neq \infty} \delta(a; h) = 1$. Also let g be an entire function and $0 < \lambda_p^{L^*}(f, h) \leq \rho_p^{L^*}(f, h) < \infty$, where p is any positive integer. Then

$$\overline{\lim}_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(\exp r^\mu)} \geq \frac{\lambda_p^{L^*}(f, h)}{\rho_p^{L^*}(f, h)},$$

where $0 < \mu < \rho_g \leq \infty$.

Proof. In view of Lemma 10, we obtain for a sequence of values of r tending to infinity that

$$\begin{aligned} \log T_h^{-1} T_{f \circ g}(r) &\geq \log T_h^{-1} T_f(\exp r^\mu), \\ \text{i.e. } \log T_h^{-1} T_{f \circ g}(r) &\geq \left(\lambda_p^{L^*}(f, h) - \varepsilon \right) \left[r^\mu + \exp^{[p-1]} L(\exp r^\mu) \right]. \end{aligned} \tag{12}$$

Also in view of Lemma 5, and for any arbitrary $\varepsilon (> 0)$, it follows for all sufficiently large values of r that

$$\begin{aligned} \log T_{P_0[h]}^{-1} T_{P_0[f]}(\exp r^\mu) &\leq \left(\rho_p^{L^*}(P_0[f], P_0[h]) + \varepsilon \right) \left[r^\mu + \exp^{[p-1]} L(\exp r^\mu) \right], \\ \text{i.e. } \log T_{P_0[h]}^{-1} T_{P_0[f]}(\exp r^\mu) &\leq \left(\rho_p^{L^*}(f, h) + \varepsilon \right) \left[r^\mu + \exp^{[p-1]} L(\exp r^\mu) \right]. \end{aligned} \tag{13}$$

Now from (12) and (13), we get for a sequence of values of r tending to infinity that

$$\frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(\exp r^\mu)} \geq \frac{\left(\lambda_p^{L^*}(f, h) - \varepsilon \right) \left[r^\mu + \exp^{[p-1]} L(\exp r^\mu) \right]}{\left(\rho_p^{L^*}(f, h) + \varepsilon \right) \left[r^\mu + \exp^{[p-1]} L(\exp r^\mu) \right]}.$$

Since $\varepsilon (> 0)$ is arbitrary, it follows from above that

$$\overline{\lim}_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(\exp r^\mu)} \geq \frac{\lambda_p^{L^*}(f, h)}{\rho_p^{L^*}(f, h)}.$$

Thus the theorem follows. □

Theorem 24. Let f be a meromorphic function, g be an entire function either of finite order or of non-zero lower order such that $\Theta(\infty; g) = \sum_{a \neq \infty} \delta_p(a; g) = 1$ or $\delta(\infty; g) = \sum_{a \neq \infty} \delta(a; g) = 1$ and h be an entire function having regular growth and non zero finite order with $\Theta(\infty; h) = \sum_{a \neq \infty} \delta_p(a; h) = 1$ or $\delta(\infty; h) = \sum_{a \neq \infty} \delta(a; h) = 1$. Let $0 < \lambda_f$ and $0 < \lambda_p^{L^*}(g, h) \leq \rho_p^{L^*}(g, h) < \infty$, where p is any positive integer. Then

$$\overline{\lim}_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[g]}(\exp r^\mu)} \geq \frac{\lambda_p^{L^*}(g, h)}{\rho_p^{L^*}(g, h)},$$

where $0 < \mu < \rho_g$.

We omit the proof of the above theorem as it can be carried out in the line of Theorem 23 and with the help of Lemma 11.

In the line of Theorem 23 and Theorem 24 respectively, one can easily prove the following two theorems and therefore their proofs are omitted.

Theorem 25. Let f be a transcendental meromorphic function of finite order or of non-zero lower order and $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; f) = 4$ and h be a transcendental entire function of regular growth having non zero finite order with $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; h) = 4$. Also let g be an entire function and $0 < \lambda_p^{L^*}(f, h) \leq \rho_p^{L^*}(f, h) < \infty$, where p is any positive integer. Then

$$\overline{\lim}_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{M[h]}^{-1} T_{M[f]}(\exp r^\mu)} \geq \frac{\lambda_p^{L^*}(f, h)}{\rho_p^{L^*}(f, h)},$$

where $0 < \mu < \rho_g \leq \infty$.

Theorem 26. Let f be a meromorphic function and g be a transcendental entire function of finite order or of non-zero lower order such that $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; g) = 4$ and h be a transcendental entire function of regular growth having non zero finite order with $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; h) = 4$. Also let $0 < \lambda_f$ and $0 < \lambda_p^{L^*}(g, h) \leq \rho_p^{L^*}(g, h) < \infty$, where p is any positive integer. Then

$$\overline{\lim}_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{M[h]}^{-1} T_{M[g]}(\exp r^\mu)} \geq \frac{\lambda_p^{L^*}(g, h)}{\rho_p^{L^*}(g, h)},$$

where $0 < \mu < \rho_g$.

Theorem 27. Let f be a meromorphic function either of finite order or of non-zero lower order such that $\Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1$ and h be an entire function having regular growth and non zero finite order with $\Theta(\infty; h) = \sum_{a \neq \infty} \delta_p(a; h) = 1$ or $\delta(\infty; h) = \sum_{a \neq \infty} \delta(a; h) = 1$. Also let g be an entire function and $0 < \lambda_p^{L^*}(f, h) \leq \rho_p^{L^*}(f, h) < \infty$, where p is any positive integer. Then

$$\overline{\lim}_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(\exp r^\mu)} \leq \frac{\rho_p^{L^*}(f, h)}{\lambda_p^{L^*}(f, h)},$$

where $\lambda_g < \mu < \infty$.

Proof. In view of Lemma 12, we obtain for a sequence of values of r tending to infinity that

$$\begin{aligned} \log T_h^{-1} T_{f \circ g}(r) &< \log T_h^{-1} T_f(\exp r^\mu), \\ \text{i.e. } \log T_h^{-1} T_{f \circ g}(r) &< \left(\rho_p^{L^*}(f, h) + \varepsilon \right) \left[r^\mu + \exp^{[p-1]} L(\exp r^\mu) \right]. \end{aligned} \quad (14)$$

Also in view of Lemma 5, and for any arbitrary $\varepsilon (> 0)$, it follows for all sufficiently large values of r that

$$\begin{aligned} \log T_{P_0[h]}^{-1} T_{P_0[f]}(\exp r^\mu) &\geq \left(\lambda_p^{L^*}(P_0[f], P_0[h]) - \varepsilon \right) \left[r^\mu + \exp^{[p-1]} L(\exp r^\mu) \right], \\ \text{i.e. } \log T_{P_0[h]}^{-1} T_{P_0[f]}(\exp r^\mu) &\geq \left(\lambda_p^{L^*}(f, h) - \varepsilon \right) \left[r^\mu + \exp^{[p-1]} L(\exp r^\mu) \right]. \end{aligned} \quad (15)$$

Now from (14) and (15), we get for a sequence of values of r tending to infinity that

$$\frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(\exp r^\mu)} < \frac{(\rho_p^{L^*}(f, h) + \varepsilon) [r^\mu + \exp^{[p-1]} L(\exp r^\mu)]}{(\lambda_p^{L^*}(f, h) - \varepsilon) [r^\mu + \exp^{[p-1]} L(\exp r^\mu)]}.$$

Since $\varepsilon (> 0)$ is arbitrary, it follows from above that

$$\lim_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(\exp r^\mu)} \leq \frac{\rho_p^{L^*}(f, h)}{\lambda_p^{L^*}(f, h)}.$$

Thus the theorem follows. □

Now we state the following theorem without its proof as it can be carried out in the line of the above theorem and with the help of Lemma 13.

Theorem 28. *Let f be a meromorphic function and g be an entire function either of finite order or of non-zero lower order such that $\Theta(\infty; g) = \sum_{a \neq \infty} \delta_p(a; g) = 1$ or $\delta(\infty; g) = \sum_{a \neq \infty} \delta(a; g) = 1$ and h be an entire function having regular growth and non zero finite order with $\Theta(\infty; h) = \sum_{a \neq \infty} \delta_p(a; h) = 1$ or $\delta(\infty; h) = \sum_{a \neq \infty} \delta(a; h) = 1$. Let $0 < \lambda_f$ and $0 < \lambda_p^{L^*}(g, h) \leq \rho_p^{L^*}(g, h) < \infty$, where p is any positive integer. Then*

$$\lim_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[g]}(\exp r^\mu)} \leq \frac{\rho_p^{L^*}(g, h)}{\lambda_p^{L^*}(g, h)},$$

where $0 < \lambda_g < \mu < \infty$.

In the line of Theorem 27 and Theorem 28 respectively, one can easily prove the following two theorems and therefore their proofs are omitted.

Theorem 29. *Let f be a transcendental meromorphic function of finite order or of non-zero lower order with $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; f) = 4$ and h be a transcendental entire function of regular growth having non zero finite order with $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; h) = 4$. Also let g be an entire function and $0 < \lambda_p^{L^*}(f, h) \leq \rho_p^{L^*}(f, h) < \infty$, where p is any positive integer. Then*

$$\lim_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{M[h]}^{-1} T_{M[f]}(\exp r^\mu)} \leq \frac{\rho_p^{L^*}(f, h)}{\lambda_p^{L^*}(f, h)},$$

where $\lambda_g < \mu < \infty$.

Theorem 30. *Let f be a meromorphic function and g be a transcendental entire function of finite order or of non-zero lower order such that $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; g) = 4$ and h be a transcendental entire function of regular growth having non zero finite order with $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; h) = 4$. Also let $0 < \lambda_f$ and $0 < \lambda_p^{L^*}(g, h) \leq \rho_p^{L^*}(g, h) < \infty$, where p is any positive integer. Then*

$$\lim_{r \rightarrow \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{M[h]}^{-1} T_{M[g]}(\exp r^\mu)} \leq \frac{\rho_p^{L^*}(g, h)}{\lambda_p^{L^*}(g, h)},$$

where $0 < \lambda_g < \mu < \infty$.

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Бісвас Т. *Прогрес у вивченні аналізу росту диференціальних поліномів і диференціальних мономів в контексті повільно зростаючих функцій* // Карпатські матем. публ. — 2018. — Т.10, №1. — С. 31–57.

Дослідження аналізу росту цілих чи мероморфних функцій, як правило, проводилися через їх характеристичну функцію Неванліни в порівнянні з тими експоненційними функціями. Але якщо потрібно порівняти темпи зростання будь-якої цілої чи мероморфної функції відносно іншої, то потрібно використовувати поняття індикаторів відносного зростання. Область дослідження в цій галузі може бути більш значимою через інтенсивні застосування теорій повільно зростаючих функцій, що фактично означає, що $L(ar) \sim L(r)$ при $r \rightarrow \infty$ для кожної додатньої константи a , тобто $\lim_{r \rightarrow \infty} \frac{L(ar)}{L(r)} = 1$, де $L \equiv L(r)$ — додатня неперервна функція, яка повільно зростає. Власне, в цій роботі ми отримали деякі результати, що залежать від властивостей відносного зростання композицій цілих і мероморфних функцій, використовуючи ідею відносного pL^* -порядку, відносного pL^* -типу, відносного pL^* -слабкого типу і диференціальних мономів, диференціальних поліномів, породжених одним з коефіцієнтів; ці результати поширюють деякі попередні результати, де pL^* є нічим іншим як слабшим припущенням на L .

Ключові слова і фрази: ціла функція, мероморфна функція, відносний pL^* порядок, відносний pL^* тип, відносний pL^* слабкий тип, ріст, диференціальний моном, диференціальний поліном, функція повільного росту.



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ON THE CONVERGENCE OF MULTIDIMENSIONAL S-FRACTIONS WITH INDEPENDENT VARIABLES

In this paper, we investigate the convergence of multidimensional S -fractions with independent variables, which are a multidimensional generalization of S -fractions. These branched continued fractions are an efficient tool for the approximation of multivariable functions, which are represented by formal multiple power series. For establishing the convergence criteria, we use the convergence continuation theorem to extend the convergence, already known for a small region, to a larger region. As a result, we have shown that the intersection of the interior of the parabola and the open disk is the domain of convergence of a multidimensional S -fraction with independent variables. Also we have shown that the interior of the parabola is the domain of convergence of a branched continued fraction, which is reciprocal to the multidimensional S -fraction with independent variables. In addition, we have obtained two new convergence criteria for S -fractions as consequences from the above mentioned results.

Key words and phrases: convergence, uniform convergence, S -fraction, multidimensional S -fraction with independent variables.

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1 INTRODUCTION

Establishing convergence criteria for the classes of functional branched continued fractions with independent variables is one of the most important tasks of their studying.

A convergence criteria have been given in [1, 2, 5] for multidimensional regular C -fractions with independent variables

$$1 + \sum_{i_1=1}^N \frac{a_{i(1)}z_{i_1}}{1} + \sum_{i_2=1}^{i_1} \frac{a_{i(2)}z_{i_2}}{1} + \sum_{i_3=1}^{i_2} \frac{a_{i(3)}z_{i_3}}{1} + \dots,$$

where the $a_{i(k)}$, $i(k) \in \mathcal{I}_k$, $k \geq 1$, are complex constants such that $a_{i(k)} \neq 0$, $i(k) \in \mathcal{I}_k$, $k \geq 1$,

$$\mathcal{I}_k = \{i(k) : i(k) = (i_1, i_2, \dots, i_k), 1 \leq i_p \leq i_{p-1}, 1 \leq p \leq k, i_0 = N\}, \quad k \geq 1,$$

denote the sets of multiindices, and where $\mathbf{z} = (z_1, z_2, \dots, z_N) \in \mathbb{C}^N$, in [8] for multidimensional g -fractions with independent variables

$$\frac{s_0}{1} + \sum_{i_1=1}^N \frac{g_{i(1)}z_{i_1}}{1} + \sum_{i_2=1}^{i_1} \frac{g_{i(2)}(1 - g_{i(1)})z_{i_2}}{1} + \sum_{i_3=1}^{i_2} \frac{g_{i(3)}(1 - g_{i(2)})z_{i_3}}{1} + \dots,$$

where the s_0 is positive constant and the $g_{i(k)}, i(k) \in \mathcal{I}_k, k \geq 1$, are real constants such that $0 < g_{i(k)} < 1, i(k) \in \mathcal{I}_k, k \geq 1$, and $\mathbf{z} \in \mathbb{C}^N$, in [6] for multidimensional associated fractions with independent variables

$$1 + \sum_{i_1=1}^N \frac{b_{i(1)}z_{i_1}}{1} + \sum_{i_2=1}^{i_1} \frac{(-1)^{\delta_{i_1,i_2}} b_{i(2)}z_{i_1}z_{i_2}}{1} + \sum_{i_3=1}^{i_2} \frac{(-1)^{\delta_{i_2,i_3}} b_{i(3)}z_{i_2}z_{i_3}}{1} + \dots,$$

where the $b_{i(k)}, i(k) \in \mathcal{I}_k, k \geq 1$, are complex constants such that $b_{i(k)} \neq 0, i(k) \in \mathcal{I}_k, k \geq 1$, and $\delta_{k,p}$ is the Kronecker delta, $1 \leq k, p \leq N, \mathbf{z} \in \mathbb{C}^N$, and in [7] for multidimensional J-fractions with independent variables

$$\sum_{i_1=1}^N \frac{-p_{i(1)}^2}{q_{i(1)} + z_{i_1}} + \sum_{i_2=1}^{i_1} \frac{-p_{i(2)}^2}{q_{i(2)} + z_{i_2}} + \sum_{i_3=1}^{i_2} \frac{-p_{i(3)}^2}{q_{i(3)} + z_{i_3}} + \dots,$$

where the $p_{i(k)}$ and $q_{i(k)}, i(k) \in \mathcal{I}_k, k \geq 1$, are complex constants such that $p_{i(k)} \neq 0, i(k) \in \mathcal{I}_k, k \geq 1$, and $\mathbf{z} \in \mathbb{C}^N$. In this paper, we investigate a convergence of multidimensional S-fraction with independent variables

$$1 + \sum_{i_1=1}^N \frac{c_{i(1)}z_{i_1}}{1} + \sum_{i_2=1}^{i_1} \frac{c_{i(2)}z_{i_2}}{1} + \sum_{i_3=1}^{i_2} \frac{c_{i(3)}z_{i_3}}{1} + \dots, \tag{1}$$

where the $c_{i(k)}, i(k) \in \mathcal{I}_k, k \geq 1$, are real constants such that $c_{i(k)} > 0, \mathbf{z} \in \mathbb{C}^N$, and reciprocal to it

$$\frac{1}{1} + \sum_{i_1=1}^N \frac{c_{i(1)}z_{i_1}}{1} + \sum_{i_2=1}^{i_1} \frac{c_{i(2)}z_{i_2}}{1} + \sum_{i_3=1}^{i_2} \frac{c_{i(3)}z_{i_3}}{1} + \dots. \tag{2}$$

We note that the multidimensional S-fraction with independent variables (1) is multidimensional generalization of S-fraction

$$1 + \frac{c_1z}{1} + \frac{c_2z}{1} + \frac{c_3z}{1} + \dots, \tag{3}$$

where the $c_k, k \geq 1$, are real constants such that $c_k > 0, k \geq 1, z \in \mathbb{C}$. A convergence result for S-fraction is as follows (see Theorem 4.58 [9, p. 136]).

Theorem 1. *Let (3) be an S-fraction and let $\mathcal{H} = \{z \in \mathbb{C} : |\arg(z)| < \pi\}$ be the complex plane cut along the negative real axis. Then the following statements hold.*

(A) *The S-fraction (3) converges to a function holomorphic in \mathcal{H} if at least one of the two series*

$$\sum_{k=1}^{\infty} \frac{c_1c_3 \dots c_{2k-1}}{c_2c_4 \dots c_{2k}}, \quad \sum_{k=1}^{\infty} \frac{c_2c_4 \dots c_{2k-2}}{c_1c_3 \dots c_{2k-1}}$$

diverges.

(B) *If the S-fraction (3) converges at a single point in \mathcal{H} , then it converges at all points in \mathcal{H} to a holomorphic function.*

(C) *A sufficient condition for an S-fraction (3) to converge to a function holomorphic in \mathcal{H} is that there exists a constant $M > 0$ such that $c_k < M, k \geq 1$.*

2 CONVERGENCE

We give two convergence criteria for multidimensional S-fraction with independent variables (1). For use in the following theorems we introduce the notation for the tails of (1):

$$Q_{i(n)}^{(n)}(\mathbf{z}) = 1, \quad i(n) \in \mathcal{I}_n, \quad n \geq 1, \quad (4)$$

$$Q_{i(k)}^{(n)}(\mathbf{z}) = 1 + \sum_{i_{k+1}=1}^{i_k} \frac{c_{i(k+1)} z_{i_{k+1}}}{1} + \sum_{i_{k+2}=1}^{i_{k+1}} \frac{c_{i(k+2)} z_{i_{k+2}}}{1} + \dots + \sum_{i_n=1}^{i_{n-1}} \frac{c_{i(n)} z_{i_n}}{1}, \quad (5)$$

where $i(k) \in \mathcal{I}_k$, $1 \leq k \leq n-1$, $n \geq 2$. It is clear that the following recurrence relations hold

$$Q_{i(k)}^{(n)}(\mathbf{z}) = 1 + \sum_{i_{k+1}=1}^{i_k} \frac{c_{i(k+1)} z_{i_{k+1}}}{Q_{i(k+1)}^{(n)}(\mathbf{z})}, \quad i(k) \in \mathcal{I}_k, \quad 1 \leq k \leq n-1, \quad n \geq 2. \quad (6)$$

Let $f_n(\mathbf{z}) = 1 + \sum_{i_1=1}^N \frac{c_{i(1)} z_{i_1}}{Q_{i(1)}^{(n)}(\mathbf{z})}$ be the n th approximant of (1), $n \geq 1$.

Theorem 2. *A multidimensional S-fraction with independent variables (1), where the $c_{i(k)}$, $i(k) \in \mathcal{I}_k$, $k \geq 2$, satisfy the conditions*

$$\sum_{i_{k+1}=1}^{i_k} c_{i(k+1)} \leq r, \quad i(k) \in \mathcal{I}_k, \quad k \geq 1, \quad (7)$$

where r is a positive number, converges to a function holomorphic in the domain

$$\mathcal{P}_{r,M} = \bigcup_{\alpha \in (-\pi/2, \pi/2)} \left\{ \mathbf{z} \in \mathbb{C}^N : |z_k| - \operatorname{Re}(z_k e^{-2i\alpha}) < \frac{\cos^2(\alpha)}{2r}, \quad |z_k| < M, \quad 1 \leq k \leq N \right\} \quad (8)$$

for every constant $M > 0$. The convergence is uniform on every compact subset of $\mathcal{P}_{r,M}$.

Proof. Let α be arbitrary number from the interval $(-\pi/2, \pi/2)$ and let n be arbitrary natural number. Using relations (6), by induction on k for arbitrary of multiindex $i(k) \in \mathcal{I}_k$ we show that the following inequalities are valid

$$\operatorname{Re}(Q_{i(k)}^{(n)}(\mathbf{z}) e^{-i\alpha}) > \frac{\cos(\alpha)}{2} > 0, \quad (9)$$

where $1 \leq k \leq n$.

It is clear that for $k = n$, $i(n) \in \mathcal{I}_n$, relations (9) hold. By induction hypothesis that (9) hold for $k = p+1$, $p \leq n-1$, $i(p+1) \in \mathcal{I}_{p+1}$, we prove (9) for $k = p$, $i(p) \in \mathcal{I}_p$. Indeed, use of relations (6) for arbitrary of multiindex $i(p) \in \mathcal{I}_p$ lead to

$$Q_{i(p)}^{(n)}(\mathbf{z}) e^{-i\alpha} = e^{-i\alpha} + \sum_{i_{p+1}=1}^{i_p} \frac{c_{i(p+1)} z_{i_{p+1}} e^{-2i\alpha}}{Q_{i(p+1)}^{(n)}(\mathbf{z}) e^{-i\alpha}}.$$

In the proof of Lemma 4.41 [9] it is shown that if $x \geq c > 0$ and $v^2 \leq 4u + 4$,

$$\min_{-\infty < y < +\infty} \operatorname{Re} \frac{u + iv}{x + iy} = -\frac{\sqrt{u^2 + v^2} - u}{2x}. \quad (10)$$

We set $u = \operatorname{Re}(c_{i(p+1)}z_{i(p+1)}e^{-2i\alpha})$, $v = \operatorname{Im}(c_{i(p+1)}z_{i(p+1)}e^{-2i\alpha})$, $x = \operatorname{Re}(Q_{i(p+1)}^{(n)}(\mathbf{z})e^{-i\alpha})$, $y = \operatorname{Im}(Q_{i(p+1)}^{(n)}(\mathbf{z})e^{-i\alpha})$. Then for the arbitrary index i_{p+1} , $1 \leq i_{p+1} \leq i_p$, it follows from (7) and (8) that

$$|c_{i(p+1)}z_{i(p+1)}e^{-2i\alpha}| - \operatorname{Re}(c_{i(p+1)}z_{i(p+1)}e^{-2i\alpha}) < \frac{\cos^2(\alpha)}{2}.$$

From this inequality it is easily shown that $v^2 \leq 4u + 4$.

Using (6)–(10) and induction hypothesis, we obtain

$$\begin{aligned} \operatorname{Re}(Q_{i(p)}^{(n)}(\mathbf{z})e^{-i\alpha}) &\geq \cos(\alpha) - \sum_{i_{p+1}=1}^{i_p} \frac{c_{i(p+1)}(|z_{i_{p+1}}| - \operatorname{Re}(z_{i_{p+1}}e^{-2i\alpha}))}{2\operatorname{Re}(Q_{i(p+1)}^{(n)}(\mathbf{z})e^{-i\alpha})} \\ &> \cos(\alpha) - \sum_{i_{p+1}=1}^{i_p} \frac{c_{i(p+1)}\cos(\alpha)}{2r} \geq \frac{\cos(\alpha)}{2} > 0. \end{aligned}$$

It follows from (9) that $Q_{i(k)}^{(n)}(\mathbf{z}) \neq 0$ for all indices. Thus, the approximants $f_n(\mathbf{z})$, $n \geq 1$, of (1) form a sequence of functions holomorphic in $\mathcal{P}_{r,M}$.

Again, let α be arbitrary number from the interval $(-\pi/2, \pi/2)$. And, let

$$\mathcal{P}_{\alpha,\sigma,r,M} = \left\{ \mathbf{z} \in \mathbb{C}^N : |z_k| - \operatorname{Re}(z_k e^{-2i\alpha}) < \frac{\sigma \cos^2(\alpha)}{2r}, |z_k| < \sigma M, 1 \leq k \leq N \right\}, \quad (11)$$

where $0 < \sigma < 1$. We set

$$c = \max_{1 \leq i_1 \leq N} c_{i_1(1)}. \quad (12)$$

Using (9), (11) and (12), for the arbitrary $\mathbf{z} \in \mathcal{P}_{\alpha,\sigma,r,M}$, $\mathcal{P}_{\alpha,\sigma,r,M} \subset \mathcal{P}_{r,M}$, we obtain for $n \geq 1$

$$|f_n(\mathbf{z})| \leq 1 + \sum_{i_1=1}^N \frac{c_{i_1(1)}|z_{i_1}|}{\operatorname{Re}(Q_{i_1(1)}^{(n)}(\mathbf{z})e^{-i\alpha})} < 1 + \sum_{i_1=1}^N \frac{2c\sigma M}{\cos(\alpha)} = C(\mathcal{P}_{\alpha,\sigma,r,M}),$$

where the constant $C(\mathcal{P}_{\alpha,\sigma,r,M})$ depends only on the domain (11), i.e. the sequence $\{f_n(\mathbf{z})\}$ is uniformly bounded in $\mathcal{P}_{\alpha,\sigma,r,M}$.

Let \mathcal{K} be an arbitrary compact subset of $\mathcal{P}_{r,M}$. Let us cover \mathcal{K} with domains of form (11). From this cover we choose the finite subcover $\mathcal{P}_{\alpha_j,\sigma_j,r,M}$, $1 \leq j \leq k$. We set

$$C(\mathcal{K}) = \max_{1 \leq j \leq k} C(\mathcal{P}_{\alpha_j,\sigma_j,r,M}).$$

Then for arbitrary $\mathbf{z} \in \mathcal{K}$ we obtain $|f_n(\mathbf{z})| \leq C(\mathcal{K})$, for $n \geq 1$, i.e. the sequence $\{f_n(\mathbf{z})\}$ is uniformly bounded on each compact subset of the domain (8).

Let $m = \max\{c, r, 1/(2MN)\}$ and let

$$\mathcal{L}_m = \left\{ \mathbf{z} \in \mathbb{R}^N : 0 < z_k < \frac{1}{4mN}, 1 \leq k \leq N \right\}.$$

Then for the arbitrary $\mathbf{z} \in \mathcal{L}_m$, $\mathcal{L}_m \subset \mathcal{P}_{r,M}$, we obtain

$$|c_{i(1)}z_{i_1}| < \frac{c}{4mN} < \frac{1}{2N}, |c_{i(k+1)}z_{i_{k+1}}| < \frac{r}{4mN} \leq \frac{1}{4i_k}, i(k) \in \mathcal{I}_k, k \geq 1.$$

It follows from Theorem 1 [4], with $g_{i(k)} = 1/2$, $i(k) \in \mathcal{I}_k$, $k \geq 1$, that (1) converges in \mathcal{L}_m . Hence by Theorem 24.2 [10, pp. 108–109] (see also Theorem 2.17 [3, p. 66]), the multidimensional S-fraction with independent variables (1) converges uniformly on compact subsets of $\mathcal{P}_{r,M}$ to a holomorphic function. \square

The following theorem can be proved in much the same way as Theorem 2 using Theorem 4 [4].

Theorem 3. *A multidimensional S-fraction with independent variables (1), where the $c_{i(k)}$, $i(k) \in \mathcal{I}_k$, $k \geq 2$, satisfy the conditions $c_{i(k)} \leq r$, $i(k) \in \mathcal{I}_k$, $k \geq 2$, where r is a positive number, converges to a function holomorphic in the domain*

$$\mathcal{D}_{r,M} = \bigcup_{\alpha \in (-\pi/2, \pi/2)} \left\{ \mathbf{z} \in \mathbb{C}^N : \sum_{k=1}^N \left(|z_k| - \operatorname{Re}(z_k e^{-2i\alpha}) \right) < \frac{\cos^2(\alpha)}{2r}, \sum_{k=1}^N |z_k| < M \right\}$$

for every constant $M > 0$. The convergence is uniform on every compact subset of $\mathcal{D}_{r,M}$.

Next, we give two convergence criteria for multidimensional S-fractions with independent variable (2). In addition to (4) and (5), for the tails of (2) we introduce the following notation:

$$Q_{i(0)}^{(0)}(\mathbf{z}) = 1, \quad Q_{i(0)}^{(n)}(\mathbf{z}) = 1 + \sum_{i_1=1}^N \frac{c_{i(1)z_{i_1}}}{1} + \sum_{i_2=1}^{i_1} \frac{c_{i(2)z_{i_2}}}{1} + \dots + \sum_{i_n=1}^{i_{n-1}} \frac{c_{i(n)z_{i_n}}}{1}, \quad n \geq 1.$$

And, thus, the n th approximant of (2) we may write as $g_n(\mathbf{z}) = 1/Q_{i(0)}^{(n-1)}(\mathbf{z})$, $n \geq 1$.

Now we shall prove the following result.

Theorem 4. *A multidimensional S-fraction with independent variables (2), where the $c_{i(k)}$, $i(k) \in \mathcal{I}_k$, $k \geq 1$, satisfy the conditions*

$$\sum_{i_k=1}^{i_{k-1}} c_{i(k)} \leq r, \quad i(k) \in \mathcal{I}_k, \quad k \geq 1, \quad (13)$$

where r is a positive number, converges to a function holomorphic in the domain

$$\mathcal{P}_r = \bigcup_{\alpha \in (-\pi/2, \pi/2)} \left\{ \mathbf{z} \in \mathbb{C}^N : |z_k| - \operatorname{Re}(z_k e^{-2i\alpha}) < \frac{\cos^2(\alpha)}{2r}, \quad 1 \leq k \leq N \right\}. \quad (14)$$

The convergence is uniform on every compact subset of \mathcal{P}_r .

Proof. Let α be arbitrary number from the interval $(-\pi/2, \pi/2)$. By analogy with (9) it is easy to prove the validity of the following inequalities

$$\operatorname{Re}(Q_{i(k)}^{(n-1)}(\mathbf{z})e^{-i\alpha}) > \frac{\cos(\alpha)}{2} > 0, \quad (15)$$

where $n \geq 1$, $0 \leq k \leq n-1$, $i(k) \in \mathcal{I}_k$, if $k \geq 1$. It follows from (15) that $Q_{i(k)}^{(n-1)}(\mathbf{z}) \neq 0$ for all indices. It means that the approximants $g_n(\mathbf{z})$, $n \geq 1$, of (2) form a sequence of functions holomorphic in \mathcal{P}_r .

Again, let α be arbitrary number from the interval $(-\pi/2, \pi/2)$. And, let

$$\mathcal{P}_{\alpha, \sigma, r} = \left\{ \mathbf{z} \in \mathbb{C}^N : |z_k| - \operatorname{Re}(z_k e^{-2i\alpha}) < \frac{\sigma \cos^2(\alpha)}{2r}, \quad 1 \leq k \leq N \right\}, \quad (16)$$

where $0 < \sigma < 1$. Using (15) for the arbitrary $\mathbf{z} \in \mathcal{P}_{\alpha,\sigma,r}$, $\mathcal{P}_{\alpha,\sigma,r} \subset \mathcal{P}_r$, we obtain for $n \geq 1$

$$|g_n(\mathbf{z})| \leq \frac{1}{\operatorname{Re}(Q_{i(0)}^{(n-1)}(\mathbf{z})e^{-i\alpha})} < \frac{2}{\cos(\alpha)} = C(\mathcal{P}_{\alpha,\sigma,r}),$$

where the constant $C(\mathcal{P}_{\alpha,\sigma,r})$ depends only on the domain (16), i.e. the sequence $\{g_n(\mathbf{z})\}$ is uniformly bounded in $\mathcal{P}_{\alpha,\sigma,r}$.

Let \mathcal{K} be an arbitrary compact subset of \mathcal{P}_r . Let us cover \mathcal{K} with domains of form (16). From this cover we choose the finite subcover $\mathcal{P}_{\alpha_1,\sigma_1,r}, \mathcal{P}_{\alpha_2,\sigma_2,r}, \dots, \mathcal{P}_{\alpha_k,\sigma_k,r}$. We set $C(\mathcal{K}) = \max_{1 \leq j \leq k} C(\mathcal{P}_{\alpha_j,\sigma_j,r})$. Then for arbitrary $\mathbf{z} \in \mathcal{K}$ we obtain $|g_n(\mathbf{z})| \leq C(\mathcal{K})$, for $n \geq 1$, i.e. the sequence $\{g_n(\mathbf{z})\}$ is uniformly bounded on each compact subset of the domain (14).

Let $\mathcal{L}_r = \left\{ \mathbf{z} \in \mathbb{R}^N : 0 < z_k < \frac{1}{4rN}, 1 \leq k \leq N \right\}$. Then from (13) for the arbitrary $\mathbf{z} \in \mathcal{L}_r$, $\mathcal{L}_r \subset \mathcal{P}_r$, we obtain

$$|c_{i(k)}z_{i_k}| < \frac{1}{4N} \leq \frac{1}{4i_{k-1}}, \quad i(k) \in \mathcal{I}_k, \quad k \geq 1.$$

It follows from Theorem 2 [4], with $g_{i(k)} = 1/2$, $i(k) \in \mathcal{I}_k, k \geq 1$, that (2) converges in \mathcal{L}_r . Hence by Theorem 24.2 [10, pp. 108–109] (see also Theorem 2.17 [3, p. 66]), the multidimensional S-fraction with independent variables (2) converges uniformly on compact subsets of \mathcal{P}_r to a holomorphic function. \square

Finally, the following theorem can be proved in much the same way as Theorem 4 using Theorem 5 [4].

Theorem 5. *A multidimensional S-fraction with independent variables (2), where the $c_{i(k)}$, $i(k) \in \mathcal{I}_k, k \geq 1$, satisfy the conditions $c_{i(k)} \leq r$, $i(k) \in \mathcal{I}_k, k \geq 1$, where r is a positive number, converges to a function holomorphic in the domain*

$$\mathcal{D}_r = \bigcup_{\alpha \in (-\pi/2, \pi/2)} \left\{ \mathbf{z} \in \mathbb{C}^N : \sum_{k=1}^N \left(|z_k| - \operatorname{Re}(z_k e^{-2i\alpha}) \right) < \frac{\cos^2(\alpha)}{2r} \right\}.$$

The convergence is uniform on every compact subset of \mathcal{D}_r .

The following two corollaries are an immediate consequences of Theorems 2 and 4 respectively.

Corollary 1. *An S-fraction (3), where the $c_k, k \geq 2$, satisfy the conditions $c_k \leq r, k \geq 2$, where r is a positive number, converges to a function holomorphic in the domain*

$$\mathcal{H}_{r,M} = \left\{ z \in \mathbb{C} : \left| \arg \left(z + \frac{1}{4r} \right) \right| < \pi, |z| < M \right\}$$

for every constant $M > 0$. The convergence is uniform on every compact subset of $\mathcal{H}_{r,M}$.

Corollary 2. *An S-fraction*

$$\frac{1}{1} + \frac{c_1 z}{1} + \frac{c_2 z}{1} + \frac{c_3 z}{1} + \dots,$$

where the $c_k, k \geq 1$, satisfy the conditions $c_k \leq r, k \geq 1$, where r is a positive number, converges to a function holomorphic in the domain

$$\mathcal{H}_r = \left\{ z \in \mathbb{C} : \left| \arg \left(z + \frac{1}{4r} \right) \right| < \pi \right\}.$$

The convergence is uniform on every compact subset of \mathcal{H}_r .

We note that, in view of Theorem 1, we conclude that Corollaries 1 and 2 give us two new convergence criteria for S-fractions.

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Досліджується збіжність багатовимірних S-дробів з нерівнозначними змінними, які є багатовимірним узагальненням S-дробів. Ці гіллясті ланцюгові дроби є ефективним інструментом для наближення функцій, заданих формальними кратними степеневими рядами. Для встановлення критеріїв збіжності використовується теорема про продовження збіжності із уже відомої малої області до більшої. У результаті показано, що перетин параболічної і кругової областей є областю збіжності багатовимірного S-дробу з нерівнозначними змінними, а параболічна область є областю збіжності гіллястого ланцюгового дробу, який є оберненим до багатовимірного S-дробу з нерівнозначними змінними. Крім того, отримано два нових критерії збіжності для S-дробів як наслідки з вище згаданих результатів.

Ключові слова і фрази: збіжність, рівномірна збіжність, S-дріб, багатовимірний S-дріб з нерівнозначними змінними.



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BOUNDARY VALUE PROBLEM SOLUTION EXISTENCE FOR LINEAR INTEGRO-DIFFERENTIAL EQUATIONS WITH MANY DELAYS

For the study of boundary value problems for delay differential equations, the contraction mapping principle and topological methods are used to obtain sufficient conditions for the existence of a solution of differential equations with a constant delay. In this paper, the ideas of the contraction mapping principle are used to obtain sufficient conditions for the existence of a solution of linear boundary value problems for integro-differential equations with many variable delays.

Smoothness properties of the solutions of such equations are studied and the definition of the boundary value problem solution is proposed. Properties of the variable delays are analyzed and functional space is obtained in which the boundary value problem is equivalent to a special integral equation. Sufficient, simple for practical verification coefficient conditions for the original equation are found under which there exists a unique solution of the boundary value problem.

Key words and phrases: boundary value problems, integro-differential equations, delay, solution existence.

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INTRODUCTION

Boundary value problems for differential and integro-differential equations with delay are an important part of the modern theory of differential-functional equations. Analytical solutions for such problems can only be found for the simplest types of equations, therefore the problem of finding approximate solutions is relevant. At the same time, it is important to study the solubility of boundary value problems with delay and properties of their solutions.

The study of the conditions for the existence of unique solutions of boundary value problems with delay using the contraction mapping principle was carried out in the papers [1, 5, 8]. Boundary value problems for differential and integro-differential equations of neutral type are investigated in [2, 3, 7] with the use of topological methods. We also note the technique of a numerical-analytic method for studying boundary value problems for differential-functional equations in papers [9, 10]. In this paper, the coefficient conditions for the existence of a solution of the boundary value problem for linear integro-differential equations with many delays, which are efficient for verification in practice, are investigated.

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1 PROBLEM STATEMENT

Let us consider the following boundary value problem

$$y''(x) = \sum_{i=0}^n \left(a_i(x)y(x - \tau_i(x)) + b_i(x)y'(x - \tau_i(x)) + \sum_{j=0}^1 \int_a^b K_{ij}(x,s)y^{(j)}(s - \tau_i(s))ds \right) + f(x), \quad (1)$$

$$y^{(j)}(x) = \varphi^{(j)}(x), \quad j = 0, 1, \quad x \in [a^*; a], \quad y(b) = \gamma, \quad (2)$$

where $\tau_0(x) = 0$ and $\tau_i(x)$, $i = \overline{1, n}$, are continuous nonnegative functions defined on $[a, b]$, $\varphi(x)$ is a continuously differentiable function given on $[a^*; a]$, $a^* = \min_{0 \leq i < n} \left\{ \inf_{x \in [a; b]} (x - \tau_i(x)) \right\}$, $\gamma \in \mathbb{R}$.

Let $a_i(x)$, $b_i(x)$, $i = \overline{0, n}$, $f(x)$ be continuous functions on $[a; b]$ and $K_{ij}(x, s)$, $i = \overline{0, n}$, $j = 0, 1$, be continuous functions of both arguments in the domain $[a, b] \times [a, b]$.

We introduce the sets of points determined by the delays $\tau_1(x), \dots, \tau_n(x)$:

$$E_i = \left\{ x_j \in [a, b] : x_j - \tau_i(x_j) = 0, \quad j = 1, 2, \dots \right\}, \quad E = \bigcup_{i=1}^n E_i.$$

Let the delays $\tau_i(x)$, $i = \overline{1, n}$, be such that the sets E_i , $i = \overline{1, n}$ are finite. We number the points of the set E in ascending order. Also, we introduce the notations:

$$J = [a^*; a], \quad I = [a, b], \quad I_1 = [a, x_1], \quad I_2 = [x_1, x_2], \quad \dots, \quad I_k = [x_{k-1}, x_k], \quad I_{k+1} = [x_k, b],$$

$$B(J \cup I) = \left\{ y(x) : y(x) \in \left(C(J \cup I) \cap (C^1(J) \cup C^1(I)) \cap \left(\bigcup_{j=1}^{k+1} C^2(I_j) \right) \right), \right. \\ \left. |y(x)| \leq P_1, \quad |y'(x)| \leq P_2 \right\},$$

where P_1, P_2 are positive constants. A function $y = y(x)$ from the space $B(J \cup I)$ is called a solution of the problem (1)–(2) if it satisfies the equation (1) on $[a; b]$ (with the possible exception of the set E) and the boundary conditions (2).

2 SOLUTION EXISTENCE

It follows from the definition of the space $B(J \cup I)$ that the solution of the problem (1)–(2) is continuously differentiable for any $x \in [a, b]$, where $y'(a)$ is the right derivative.

Let us introduce a norm in the space $B(J \cup I)$:

$$\|y\|_B = \max \left\{ \frac{8}{(b-a)^2} \max_{x \in J \cup I} |y(x)|, \quad \frac{2}{b-a} \max \left(\max_{x \in J} |y'(x)|, \max_{x \in I} |y'(x)| \right) \right\}.$$

The space $B(J \cup I)$ with this norm is a Banach space. The boundary value problem (1)–(2) is equivalent to the following integral equation [5, 7]:

$$y(x) = \int_{a^*}^b \left[\sum_{i=0}^n \left(a_i(s) y(s - \tau_i(s)) + b_i(s) y'(s - \tau_i(s)) + \sum_{j=0}^1 \int_a^b K_{ij}(s, \xi) y^{(j)}(\xi - \tau_i(\xi)) d\xi \right) \right] \overline{G}(x, s) ds + l(x), \quad x \in J \cup I, \tag{3}$$

$$\overline{G}(x, s) = \begin{cases} G(x, s), & x, s \in I, \\ 0, & \text{otherwise,} \end{cases} \quad l(x) = \begin{cases} \varphi(x), & x \in J, \\ \frac{\gamma - \varphi(a)}{b-a}(x - a) + \varphi(a), & x \in I, \end{cases}$$

where $G(x, s)$ is the Green function of the following boundary value problem $y''(x) = 0$, $x \in I$, $y(a) = y(b) = 0$. We define the operator T in the space $B(J \cup I)$ in the following way

$$(Ty)(x) = \int_{a^*}^b \left[\sum_{i=0}^n \left(a_i(s) y(s - \tau_i(s)) + b_i(s) y'(s - \tau_i(s)) + \sum_{j=0}^1 \int_a^b K_{ij}(s, \xi) y^{(j)}(\xi - \tau_i(\xi)) d\xi \right) \right] \overline{G}(x, s) ds + l(x), \quad x \in J \cup I. \tag{4}$$

$$(Ty)'(x) = \int_{a^*}^b \left[\sum_{i=0}^n \left(a_i(s) y(s - \tau_i(s)) + b_i(s) y'(s - \tau_i(s)) + \sum_{j=0}^1 \int_a^b K_{ij}(s, \xi) y^{(j)}(\xi - \tau_i(\xi)) d\xi \right) \right] \overline{G}'_x(x, s) ds + \frac{\gamma - \varphi(a)}{b-a}, \quad x \in J \cup I. \tag{5}$$

Let the coefficients in the equation (1) be such that the following inequalities are true $|a_i(x)| \leq A_i$, $|b_i(x)| \leq B_i$, $|K_{ij}(x, s)| \leq \overline{K}_{ij}$, $i = \overline{0, n}$, $j = \overline{0, 1}$, $|f(x)| \leq F$, $x \in [a; b]$. We denote $P = \sum_{i=0}^n \left(A_i P_1 + B_i P_2 + (b - a) \sum_{j=0}^1 \overline{K}_{ij} P_{j+1} \right) + F$, where P_1, P_2 are the positive constants which are included in the definition of space $B(J \cup I)$.

Theorem 1. *Let the following conditions hold:*

- 1) $\max \left\{ \max_{x \in J} |\varphi(x)|, \frac{(b-a)^2}{8} P + \max(|\varphi(a)|, |\gamma|) \right\} \leq P_1$,
- 2) $\max \left\{ \max_{x \in J} |\varphi'(x)|, \frac{b-a}{2} P + \left| \frac{\gamma - \varphi(a)}{b-a} \right| \right\} \leq P_2$,
- 3) $\frac{(b-a)^2}{8} \sum_{i=0}^n \left(A_i + (b - a) \overline{K}_{i0} \right) + \frac{b-a}{2} \sum_{i=0}^n \left(B_i + (b - a) \overline{K}_{i1} \right) < 1$.

Then there exists a unique solution of the problem (1)–(2) in $B(J \cup I)$.

Proof. Based on Green’s function

$$G(x, s) = \begin{cases} \frac{(s-a)(x-b)}{b-a}, & a \leq s \leq x \leq b, \\ \frac{(x-a)(s-b)}{b-a}, & a \leq x \leq s \leq b, \end{cases}$$

we obtain the following estimates:

$$\int_a^b |G(x, s)| ds \leq \frac{(b-a)^2}{8}, \quad \int_a^b |G'_x(x, s)| ds \leq \frac{b-a}{2}. \quad (6)$$

When the conditions 1)—2) and the inequalities (6) are true, the operator T maps the space $B(J \cup I)$ onto itself. Let $y_1, y_2 \in B(J \cup I)$. Considering the estimates (6), we get

$$\begin{aligned} |(Ty_1)(x) - (Ty_2)(x)| &= \left| \int_{a^*}^b \left[\sum_{i=0}^n \left(a_i(s) (y_1(s - \tau_i(s)) - y_2(s - \tau_i(s))) \right. \right. \right. \\ &\quad \left. \left. + b_i(s) (y'_1(s - \tau_i(s)) - y'_2(s - \tau_i(s))) \right. \right. \\ &\quad \left. \left. + \sum_{j=0}^1 \int_a^b K_{ij}(s, \xi) (y_1^{(j)}(\xi - \tau_i(\xi)) - y_2^{(j)}(\xi - \tau_i(\xi))) d\xi \right) \right] \bar{G}(x, s) ds \Big| \\ &\leq \int_{a^*}^b \left[\sum_{i=0}^n \left(A_i |y_1(s - \tau_i(s)) - y_2(s - \tau_i(s))| + B_i |y'_1(s - \tau_i(s)) - y'_2(s - \tau_i(s))| \right. \right. \\ &\quad \left. \left. + \sum_{j=0}^1 \int_a^b |K_{ij}(s, \xi)| |y_1^{(j)}(\xi - \tau_i(\xi)) - y_2^{(j)}(\xi - \tau_i(\xi))| d\xi \right) \right] |\bar{G}(x, s)| ds \\ &\leq \int_{a^*}^b \left[\sum_{i=0}^n \left(A_i \max_{s \in J \cup I} |y_1 - y_2| + B_i \max \left\{ \max_{s \in I} |y'_1 - y'_2|, \max_{s \in J} |y'_1 - y'_2| \right\} \right. \right. \\ &\quad \left. \left. + \int_a^b \bar{K}_{i0} \max_{s \in J \cup I} |y_1 - y_2| d\xi + \int_a^b \bar{K}_{i1} \max \left\{ \max_{s \in I} |y'_1 - y'_2|, \max_{s \in J} |y'_1 - y'_2| \right\} d\xi \right) \right] |\bar{G}(x, s)| ds \\ &= \int_{a^*}^b \left[\frac{(b-a)^2}{8} \frac{8}{(b-a)^2} \max_{s \in J \cup I} |y_1 - y_2| \sum_{i=0}^n A_i \right. \\ &\quad \left. + \frac{b-a}{2} \frac{2}{b-a} \max \left\{ \max_{s \in I} |y'_1 - y'_2|, \max_{s \in J} |y'_1 - y'_2| \right\} \sum_{i=0}^n B_i \right. \\ &\quad \left. + \frac{(b-a)^2}{8} \frac{8}{(b-a)^2} \max_{s \in J \cup I} |y_1 - y_2| (b-a) \sum_{i=0}^n \bar{K}_{i0} \right. \\ &\quad \left. + \frac{b-a}{2} \frac{2}{b-a} \max \left\{ \max_{s \in I} |y'_1 - y'_2|, \max_{s \in J} |y'_1 - y'_2| \right\} (b-a) \sum_{i=0}^n \bar{K}_{i1} \right] |\bar{G}(x, s)| ds \\ &\leq \frac{(b-a)^2}{8} \left[\frac{(b-a)^2}{8} \max_{s \in J \cup I} |y_1 - y_2| \left(\sum_{i=0}^n A_i + (b-a) \sum_{i=0}^n \bar{K}_{i0} \right) \right. \\ &\quad \left. + \frac{b-a}{2} \frac{2}{b-a} \max \left\{ \max_{s \in I} |y'_1 - y'_2|, \max_{s \in J} |y'_1 - y'_2| \right\} \left(\sum_{i=0}^n B_i + (b-a) \sum_{i=0}^n \bar{K}_{i1} \right) \right] \\ &\leq \|y_1 - y_2\|_B \frac{(b-a)^2}{8} \left[\frac{(b-a)^2}{8} \sum_{i=0}^n \left(A_i + (b-a) \bar{K}_{i0} \right) + \frac{b-a}{2} \sum_{i=0}^n \left(B_i + (b-a) \bar{K}_{i1} \right) \right]. \end{aligned}$$

$$\begin{aligned} |(Ty'_1)(x) - (Ty'_2)(x)| &\leq \|y_1 - y_2\|_B \frac{b-a}{2} \left[\frac{(b-a)^2}{8} \sum_{i=0}^n \left(A_i + (b-a)\bar{K}_{i0} \right) \right. \\ &\quad \left. + \frac{b-a}{2} \sum_{i=0}^n \left(B_i + (b-a)\bar{K}_{i1} \right) \right]. \end{aligned}$$

Based on the obtained estimates, we have

$$\begin{aligned} &\max_{x \in J \cup I} |(Ty_1)(x) - (Ty_2)(x)| \\ &\leq \|y_1 - y_2\|_B \frac{(b-a)^2}{8} \left[\frac{(b-a)^2}{8} \sum_{i=0}^n \left(A_i + (b-a)\bar{K}_{i0} \right) + \frac{b-a}{2} \sum_{i=0}^n \left(B_i + (b-a)\bar{K}_{i1} \right) \right], \\ &\max \left\{ \max_{x \in J} |(Ty'_1)(x) - (Ty'_2)(x)|, \max_{x \in I} |(Ty'_1)(x) - (Ty'_2)(x)| \right\} \\ &\leq \|y_1 - y_2\|_B \frac{b-a}{2} \left[\frac{(b-a)^2}{8} \sum_{i=0}^n \left(A_i + (b-a)\bar{K}_{i0} \right) + \frac{b-a}{2} \sum_{i=0}^n \left(B_i + (b-a)\bar{K}_{i1} \right) \right]. \end{aligned}$$

We multiply the first inequality by $\frac{8}{(b-a)^2}$ and the second one on $\frac{2}{b-a}$:

$$\begin{aligned} &\frac{8}{(b-a)^2} \max_{x \in J \cup I} |(Ty_1)(x) - (Ty_2)(x)| \\ &\leq \|y_1 - y_2\|_B \left[\frac{(b-a)^2}{8} \sum_{i=0}^n \left(A_i + (b-a)\bar{K}_{i0} \right) + \frac{b-a}{2} \sum_{i=0}^n \left(B_i + (b-a)\bar{K}_{i1} \right) \right], \\ &\frac{2}{b-a} \max \left\{ \max_{x \in J} |(Ty'_1)(x) - (Ty'_2)(x)|, \max_{x \in I} |(Ty'_1)(x) - (Ty'_2)(x)| \right\} \\ &\leq \|y_1 - y_2\|_B \left[\frac{(b-a)^2}{8} \sum_{i=0}^n \left(A_i + (b-a)\bar{K}_{i0} \right) + \frac{b-a}{2} \sum_{i=0}^n \left(B_i + (b-a)\bar{K}_{i1} \right) \right]. \end{aligned}$$

Given the resulting inequalities, we get

$$\begin{aligned} &\max \left\{ \frac{8}{(b-a)^2} \max_{x \in J \cup I} |(Ty_1)(x) - (Ty_2)(x)|, \right. \\ &\quad \left. \frac{2}{b-a} \max \left\{ \max_{x \in J} |(Ty'_1)(x) - (Ty'_2)(x)|, \max_{x \in I} |(Ty'_1)(x) - (Ty'_2)(x)| \right\} \right\} \\ &\leq \|y_1 - y_2\|_B \left[\frac{(b-a)^2}{8} \sum_{i=0}^n \left(A_i + (b-a)\bar{K}_{i0} \right) + \frac{b-a}{2} \sum_{i=0}^n \left(B_i + (b-a)\bar{K}_{i1} \right) \right]. \end{aligned}$$

From the definition of the norm in the space $B(J \cup I)$ we have:

$$\begin{aligned} &\| (Ty_1)(x) - (Ty_2)(x) \|_B \\ &\leq \|y_1 - y_2\|_B \left[\frac{(b-a)^2}{8} \sum_{i=0}^n \left(A_i + (b-a)\bar{K}_{i0} \right) + \frac{b-a}{2} \sum_{i=0}^n \left(B_i + (b-a)\bar{K}_{i1} \right) \right]. \quad (7) \end{aligned}$$

The inequality (7) and the condition 3) imply that the operator T is a contraction in $B(J \cup I)$ and it has a single fixed point in this space [6], therefore the boundary value problem (1)–(2) has a unique solution $y(x) \in B(J \cup I)$. The proof is complete. \square

Remark. An efficient algorithm for finding an approximate solution of the boundary value problem (1)–(2) is the spline approximation method, using cubic splines with defect 2, which is considered in the paper [4].

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Для дослідження крайових задач для диференціальних рівнянь із запізненням застосовуються методи стислих відображень та топологічні методи для одержання достатніх умов існування розв'язку диференціальних рівнянь зі сталим запізненням. У даній роботі використовуються ідеї методу стислих відображень для одержання достатніх умов існування розв'язку лінійних крайових задач для інтегро-диференціальних рівнянь із багатьма змінними запізненнями.

Досліджено властивості гладкості розв'язків таких рівнянь та запропоновано означення розв'язку крайової задачі. Проаналізовано властивості змінних запізнень і одержано функціональний простір, в якому крайова задача еквівалентна спеціальному інтегральному рівнянню. Знайдено легкі для практичної перевірки достатні коефіцієнтні умови на вихідне рівняння, при виконанні яких існує єдиний розв'язок крайової задачі.

Ключові слова і фрази: крайові задачі, інтегро-диференціальні рівняння, запізнення, існування розв'язку.

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A GENERALIZATION OF A LOCALIZATION PROPERTY OF BESOV SPACES

The notion of a localization property of Besov spaces is introduced by G. Bourdaud, where he has provided that the Besov spaces $B_{p,q}^s(\mathbb{R}^n)$, with $s \in \mathbb{R}$ and $p, q \in [1, +\infty]$ such that $p \neq q$, are not localizable in the ℓ^p norm. Further, he has provided that the Besov spaces $B_{p,q}^s$ are embedded into localized Besov spaces $(B_{p,q}^s)_{\ell^p}$ (i.e., $B_{p,q}^s \hookrightarrow (B_{p,q}^s)_{\ell^p}$, for $p \geq q$). Also, he has provided that the localized Besov spaces $(B_{p,q}^s)_{\ell^p}$ are embedded into the Besov spaces $B_{p,q}^s$ (i.e., $(B_{p,q}^s)_{\ell^p} \hookrightarrow B_{p,q}^s$, for $p \leq q$). In particular, $B_{p,p}^s$ is localizable in the ℓ^p norm, where ℓ^p is the space of sequences $(a_k)_k$ such that $\|(a_k)_k\|_{\ell^p} < \infty$. In this paper, we generalize the Bourdaud theorem of a localization property of Besov spaces $B_{p,q}^s(\mathbb{R}^n)$ on the ℓ^r space, where $r \in [1, +\infty]$. More precisely, we show that any Besov space $B_{p,q}^s$ is embedded into the localized Besov space $(B_{p,q}^s)_{\ell^r}$ (i.e., $B_{p,q}^s \hookrightarrow (B_{p,q}^s)_{\ell^r}$, for $r \geq \max(p, q)$). Also we show that any localized Besov space $(B_{p,q}^s)_{\ell^r}$ is embedded into the Besov space $B_{p,q}^s$ (i.e., $(B_{p,q}^s)_{\ell^r} \hookrightarrow B_{p,q}^s$, for $r \leq \min(p, q)$). Finally, we show that the Lizorkin-Triebel spaces $F_{p,q}^s(\mathbb{R}^n)$, where $s \in \mathbb{R}$ and $p \in [1, +\infty)$ and $q \in [1, +\infty]$ are localizable in the ℓ^p norm (i.e., $F_{p,q}^s = (F_{p,q}^s)_{\ell^p}$).

Key words and phrases: Besov spaces, Lizorkin-Triebel spaces, localization property.

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INTRODUCTION

Functional calculus is one of the basic theory in functional analysis [5]. It has enabled to study function-analytic in topological (in particular, normed) spaces of functions. For instance, several authors such as Peetre [7], Dahlberg [4], Marcus and Mizel [6] have studied functional calculus in certain Sobolev and Besov spaces. In particular, Bourdaud [1, 2] have established a way of functional calculus in localized Besov spaces. More precisely, in [1] he has proved the following result.

Theorem 1. *Let $p, q \in [1, +\infty]$, $s \in \mathbb{R}$, $B_{p,q}^s$ and $(B_{p,q}^s)_{\ell^p}$ are respectively the Besov and localized Besov spaces. Then*

- (i) $B_{p,q}^s \hookrightarrow (B_{p,q}^s)_{\ell^p}$, for $p \geq q$,
- (ii) $(B_{p,q}^s)_{\ell^p} \hookrightarrow B_{p,q}^s$ for $p \leq q$.

In particular, $B_{p,p}^s$ is localizable in the ℓ^p norm, where ℓ^p is the space of sequences $(a_k)_k$ such that $\|(a_k)_k\|_{\ell^p} = (\sum_{k=0}^{\infty} |a_k|^p)^{\frac{1}{p}} < \infty$.

In this paper, we generalize this result by proving that it is valid for any ℓ^r space, where $r \in [1, +\infty]$. This paper is organized as follows. In section 1, we recall basic concepts of Besov and Lizorkin-Triebel spaces, the decomposition of Littlewood-Paley, and some notations that will be needed throughout this paper. In section 2, we give a generalization of Bourdaud theorem of a localization property of Besov spaces on the ℓ^r space, where $r \in [1, +\infty]$. Also, we show that the Lizorkin-Triebel spaces are localizable in the ℓ^p norm. Finally, we present some conclusions and discuss future research in section 3.

1 PRELIMINARIES AND NOTATIONS

This section contains the basic definitions and notations that will be needed throughout this paper.

1.1 Notations

We note (e_1, \dots, e_n) the canonical basis of \mathbb{R}^n , $x \cdot y = x_1 y_1 + \dots + x_n y_n$ the scalar product in \mathbb{R}^n , and for $\alpha \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + \dots + \alpha_n$, $\frac{\partial^{|\alpha|} f}{\partial^{|\alpha|} x_1 \dots \partial^{|\alpha|} x_n}$ the partial derivative of the function f is denoted by $\partial^\alpha f$.

If $f : \mathbb{R}^n \rightarrow \mathbb{C}$, the support of f denoted by $\text{supp} f$. $\mathcal{D}(\mathbb{R}^n)$ is the space of test functions, i.e. of smooth functions which have compact support, $\mathcal{D}'(\mathbb{R}^n)$ is the dual of $\mathcal{D}(\mathbb{R}^n)$. $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space of functions $C^\infty(\mathbb{R}^n)$ rapidly decreasing on \mathbb{R}^n , the dual $\mathcal{S}'(\mathbb{R}^n)$ is the space of tempered distributions.

If $f \in \mathcal{S}(\mathbb{R}^n)$, then its Fourier transform defined by

$$\mathcal{F}(f(x))(\xi) = \int_{\mathbb{R}^n} \exp(-ix \cdot \xi) f(x) dx$$

and its inverse Fourier transform defined by

$$\mathcal{F}^{-1}(\widehat{f}(\xi))(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \exp(ix \cdot \xi) \widehat{f}(\xi) d\xi.$$

Let A_1 and A_2 be two spaces, we say that $A_1 \hookrightarrow A_2$ if there exists $c > 0$ such that $\|\cdot\|_{A_2} \leq c \|\cdot\|_{A_1}$. Let p' be the conjugate exponent of p , $\frac{1}{p} + \frac{1}{p'} = 1$ where $p \in [1, +\infty]$.

Let $k \in \mathbb{Z}^n$; τ_k is the translation operator defined by $\tau_k f(\cdot) = f(\cdot - k)$; L^p is the space of the measurable functions f such that $\|f\|_{L^p} = (\int_{\mathbb{R}^n} |f(x)|^p dx)^{\frac{1}{p}} < \infty$; ℓ^q is the space of sequences $(a_k)_k$ such that $\|(a_k)\|_{\ell^q} = (\sum_{k=0}^{\infty} |a_k|^q)^{\frac{1}{q}} < \infty$.

Let $0 < p \leq \infty$, $0 < q \leq \infty$, so

$$\|f_k\|_{\ell^q(L^p)} = \left(\sum_{k=0}^{\infty} \|f_k(x)\|_p^q \right)^{\frac{1}{q}} < \infty, \quad \|f_k\|_{L^p(\ell^q)} = \left\| \left(\sum_{k=0}^{\infty} |f_k(x)|^q \right)^{\frac{1}{q}} \right\|_p < \infty.$$

1.2 The decomposition of Littlewood-Paley

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$, which satisfy the conditions:

- (i) $\text{supp } \varphi \subset \{\xi \in \mathbb{R}^n : 1 \leq |\xi| \leq 3\}$,
- (ii) $\varphi(\xi) > 0$, for $1 \leq |\xi| \leq 3$,
- (iii) $\sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1$, for $\xi \in \mathbb{R}^n \setminus \{0\}$.

The construction of φ does not pose any difficulty, see for example [3]. We put $\varphi(\xi) = 1 - \sum_{j=1}^{\infty} \varphi(2^{-j}\xi)$, then it follows that the function $\varphi \in \mathcal{C}^{\infty}(\mathbb{R}^n)$, such that $\text{supp } \varphi \subset \{\xi \in \mathbb{R}^n : |\xi| \leq 3\}$. In the following, we fix the partition of the unit and we obtain:

$$\varphi(\xi) + \sum_{j=1}^{\infty} \varphi(2^{-j}\xi) = 1 \quad (\text{for all } \xi \in \mathbb{R}^n).$$

To this partition we associate a sequence of convolution operators $\Delta_j : \mathcal{S}' \rightarrow \mathcal{C}^{\infty}$, defined by $\mathcal{F}(\Delta_j f)(\xi) = \varphi(2^{-j}\xi)\widehat{f}(\xi)$, for $j = 1, 2, \dots$ and $\mathcal{F}(\Delta_0 f)(\xi) = \varphi(\xi)\widehat{f}(\xi)$. Also, we define the operators Q_k by $\mathcal{F}(Q_k f)(\xi) = \varphi(2^{-k}\xi)\widehat{f}(\xi)$, $k = 1, 2, \dots$, for all $f \in \mathcal{S}'$, the decomposition of f of the Littlewood-Paley type given by

$$f = \sum_{j \geq 0} \Delta_j f. \tag{1}$$

The series (1) converges in the sense of tempered distributions. The series (1) can be written as

$$f = Q_k f + \sum_{j \geq k+1} \Delta_j f,$$

This formula is valid for any $f \in \mathcal{S}'$ and $k \in \mathbb{N}$, such that $Q_k f = \sum_{j \leq k} \Delta_j f$.

Definition 1 ([10]). Let $f \in \mathcal{S}'$ and $a > 0$. We define the maximal operators associated to the Δ_k and Q_k by

$$\Delta_k^{*,a} f(x) = \sup_{y \in \mathbb{R}^n} \frac{|\Delta_k f(x-y)|}{(1+2^k|y|)^a} \quad \text{and} \quad Q_k^{*,a} f(x) = \sup_{y \in \mathbb{R}^n} \frac{|Q_k f(x-y)|}{(1+2^k|y|)^a}.$$

Definition 2 ([8]). Let $s \in \mathbb{R}$, $p, q \in [1, +\infty]$. The Besov space $B_{p,q}^s(\mathbb{R}^n)$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfying

$$\|f\|_{B_{p,q}^s(\mathbb{R}^n)} = \begin{cases} (\sum_{j \geq 0} (2^{sj} \|\Delta_j f\|_p)^q)^{\frac{1}{q}} < +\infty, & \text{for } q \neq \infty, \\ \sup_{j \geq 0} (2^{sj} \|\Delta_j f\|_p) < +\infty, & \text{for } q = \infty. \end{cases} \tag{2}$$

Definition 3 ([8]). Let $s \in \mathbb{R}$, $p \in [1, +\infty[$ and $q \in [1, +\infty]$. The Lizorkin-Triebel space $F_{p,q}^s(\mathbb{R}^n)$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfying

$$\|f\|_{F_{p,q}^s(\mathbb{R}^n)} = \begin{cases} \|(\sum_{j \geq 0} (2^{sj} |\Delta_j f|)^q)^{\frac{1}{q}}\|_p < +\infty, & \text{for } q \neq \infty. \\ \|\sup_{j \geq 0} (2^{sj} |\Delta_j f|)\|_p < +\infty, & \text{for } q = \infty. \end{cases} \tag{3}$$

Remark 1. In the formula (2) (resp. (3)), we can replace Δ_j by $\Delta_j^{*,a}$ with $a > \frac{n}{p}$ (resp. $a > \frac{n}{\min(p,q)}$), and we obtain an equivalent norm in $B_{p,q}^s(\mathbb{R}^n)$ (resp. $F_{p,q}^s(\mathbb{R}^n)$).

For more details, see Peetre [7] and Triebel [10].

Proposition 1 ([2]). Let $s \in \mathbb{R}$.

(i) For all $\gamma > 1$ there exists $c > 0$ such that for any sequence of functions $(f_j)_{j \geq 0}$, where $\text{supp } \mathcal{F} f_j \subset \{\xi : \gamma^{-1}2^j \leq |\xi| \leq \gamma 2^j\}$, we have

$$\left\| \sum_{j=0}^{\infty} f_j \right\|_{B_{p,q}^s} \leq c \left(\sum_{j=0}^{\infty} 2^{sjq} \|f_j\|_p^q \right)^{\frac{1}{q}}.$$

(ii) For all $a > 1$ there exists $c > 0$ such that for any sequence of functions $(f_j)_{j \in \mathbb{N}}$, where $\text{supp } \mathcal{F} f_j \subset \{\xi : a^{-1}2^j \leq |\xi| \leq a 2^j\}$, we have

$$\left\| \sum_{j=0}^{\infty} f_j \right\|_{F_{p,q}^s} \leq c \left\| \left(\sum_{j=0}^{\infty} 2^{sjq} |f_j|^q \right)^{\frac{1}{q}} \right\|_p.$$

2 LOCALIZATION OF BESOV SPACES

In this section, we give a generalization of Bourdaud theorem of a localization property of Besov spaces on the ℓ^r space, where $r \in [1, +\infty]$. Also, we show that the Lizorkin-Triebel spaces are localizable in the ℓ^p norm. We start with these important concepts.

Let E be a Banach space of distributions. We associate on the space E the following hypothesis.

- 1) Translation invariance; if we denote τ_x the operator given by $\tau_x f(t) = f(x - t)$, then τ_x is an isometric of E ;
- 2) Localization invariance; for all $f \in E$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$, we have that $\varphi f \in E$.

Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$. The notion of localized is defined by $f_x = \tau_x \varphi \cdot f$, it follows immediately from the hypothesis 1) and 2) that the family $(f_x)_{x \in \mathbb{R}^n}$ is bounded in E . We consider the set A as the class of all the functions $\varphi \in \mathcal{D}(\mathbb{R}^n)$ satisfying

$$\sum_{k \in \mathbb{Z}^n} \varphi(x - k) = 1 \quad \text{for all } x \in \mathbb{R}^n.$$

Definition 4 ([1]). *Let E be a Banach space of distributions, E is localizable in the ℓ^p norm ($1 \leq p \leq \infty$), if there exist $\varphi \in A$ and a constant $c \geq 1$, such that*

$$c^{-1} \|f\|_E \leq \left(\sum_{k \in \mathbb{Z}^n} \|\tau_k \varphi \cdot f\|_E^p \right)^{\frac{1}{p}} \leq c \|f\|_E,$$

i.e. $E = (E)_{\ell^p}$, we denote by $(E)_{\ell^p}$ the distribution space of u such that

$$\|u\|_{(E)_{\ell^p}} = \|(\|\tau_k \varphi \cdot u\|_E)_{k \in \mathbb{Z}^n}\|_{\ell^p} < \infty.$$

Proposition 2 ([1]). *Let \mathcal{S} be the Schwartz space, if the function $\theta \in \mathcal{S}$ is not null on the support of φ , then we have*

$$\|u\|_{(E)_{\ell^p}} \sim \|(\|\tau_k \theta \cdot u\|_E)_{k \in \mathbb{Z}^n}\|_{\ell^p}.$$

Proposition 3 ([1]). *Let $B_{p,q}^s$ be a Besov space, and N be a natural number fulfill $N > s$, and $\lambda, \mu \in \mathcal{S}$, such that*

$$(i) \quad \mu(\xi) \neq 0, \text{ for } |\xi| \leq 3,$$

$$(ii) \quad \lambda(\xi) \neq 0, \text{ for } 1 \leq |\xi| \leq 3 \text{ and } \lambda^{(\alpha)}(0) = 0 \text{ for } |\alpha| < N.$$

We denote by $L_j (j \geq 1)$ the respective symbol operators $\lambda(2^{-j}\xi)$ and by L_0 the symbol operator $\mu(\xi)$, therefore

$$\|u\|_{B_{p,q}^s} \sim \|(2^{js} \|L_j u\|_p)_{j \in \mathbb{N}}\|_{\ell^q}.$$

In the following theorem we give a generalization of Bourdaud theorem of a localization property of Besov spaces on the ℓ^r spaces, by using Proposition 2 and Proposition 3.

Theorem 2. *Let $p, q, r \in [1, +\infty]$, $s \in \mathbb{R}$, and $B_{p,q}^s$ and $(B_{p,q}^s)_{\ell^r}$ are respectively the Besov and localized Besov spaces. Then*

$$(i) \quad B_{p,q}^s \hookrightarrow (B_{p,q}^s)_{\ell^r} \text{ for } r \geq \max(p, q),$$

(ii) $(B_{p,q}^s)_{\ell^r} \hookrightarrow B_{p,q}^s$ for $r \leq \min(p, q)$.

In particular, $B_{p,p}^s$ space is localizable in the ℓ^p norm.

Proof. (i) We will show that

$$\|u\|_{(B_{p,q}^s)_{\ell^r}} \leq c \|u\|_{B_{p,q}^s} \quad \text{for } c > 0.$$

By Proposition 1, it follows that

$$\left\| \sum_{j \geq 0} \tau_k \theta \cdot \Delta_j u \right\|_{B_{p,q}^s} \leq c \left(\sum_{j \geq 0} 2^{sjq} \|\tau_k \theta \cdot \Delta_j u\|_p^q \right)^{\frac{1}{q}}.$$

This implies that, $\|\tau_k \theta \cdot u\|_{B_{p,q}^s}^r \leq c \left(\sum_{j=0}^{\infty} 2^{sjq} \|\tau_k \theta \cdot \Delta_j u\|_p^q \right)^{\frac{r}{q}}$. Then it holds that

$$\left(\sum_{k \in \mathbb{Z}^n} \|\tau_k \theta \cdot u\|_{B_{p,q}^s}^r \right)^{\frac{1}{r}} \leq c \left(\sum_{k \in \mathbb{Z}^n} \left(\sum_{j=0}^{\infty} 2^{sjq} \|\tau_k \theta \cdot \Delta_j u\|_p^q \right)^{\frac{r}{q}} \right)^{\frac{1}{r}}.$$

Consequently

$$\left(\sum_{k \in \mathbb{Z}^n} \|\tau_k \theta \cdot u\|_{B_{p,q}^s}^r \right)^{\frac{1}{r}} \leq c \left(\|(2^{sj} \|\tau_k \theta \cdot \Delta_j u\|_p)_{k \in \mathbb{Z}^n}\|_{\ell^q(\ell^r)} \right). \quad (4)$$

Since, $r \geq \max(p, q)$ implies that $q \leq r$. Then from Minkowski inequality we have

$$\|(2^{sj} \|\tau_k \theta \cdot \Delta_j u\|_p)_{k \in \mathbb{Z}^n}\|_{\ell^r(\ell^q)} \leq c \|(2^{sj} \|\tau_k \theta \cdot \Delta_j u\|_p)_{k \in \mathbb{Z}^n}\|_{\ell^q(\ell^r)}.$$

So, we can see that the inequality (4) becomes as follows

$$\left(\sum_{k \in \mathbb{Z}^n} \|\tau_k \theta \cdot u\|_{B_{p,q}^s}^r \right)^{\frac{1}{r}} \leq c \left(\|(2^{sj} \|\tau_k \theta \cdot \Delta_j u\|_p)_{k \in \mathbb{Z}^n}\|_{\ell^q(\ell^r)} \right).$$

Consequently $\left(\sum_{k \in \mathbb{Z}^n} \|\tau_k \theta \cdot u\|_{B_{p,q}^s}^r \right)^{\frac{1}{r}} \leq c \left(\sum_{j=0}^{\infty} 2^{sjq} \left(\sum_{k \in \mathbb{Z}^n} \|\tau_k \theta \cdot \Delta_j u\|_p^q \right)^{\frac{r}{q}} \right)^{\frac{1}{q}}$. Therefore,

$$\left(\sum_{k \in \mathbb{Z}^n} \|\tau_k \theta \cdot u\|_{B_{p,q}^s}^r \right)^{\frac{1}{r}} \leq c \left(\sum_{j=0}^{\infty} 2^{sjq} (\|\tau_k \theta \cdot \Delta_j u\|_{\ell^p(L^p)})^q \right)^{\frac{1}{q}}. \quad (5)$$

Also, we have $r \geq \max(p, q)$ implies that $p \leq r$, i.e. $\ell^p \hookrightarrow \ell^r$, it follows that $\ell^p(L^p) \hookrightarrow \ell^r(L^p)$. Consequently $\|(\tau_k \theta \cdot \Delta_j u)_{k \in \mathbb{Z}^n}\|_{\ell^r(L^p)} \leq c \|(\tau_k \theta \cdot \Delta_j u)_{k \in \mathbb{Z}^n}\|_{\ell^p(L^p)}$. So, we can see that the inequality (5) becomes as follows $\left(\sum_{k \in \mathbb{Z}^n} \|\tau_k \theta \cdot u\|_{B_{p,q}^s}^r \right)^{\frac{1}{r}} \leq c \left(\sum_{j=0}^{\infty} 2^{sjq} (\|\tau_k \theta \cdot \Delta_j u\|_{\ell^p(L^p)})^q \right)^{\frac{1}{q}}$. Since L^p is a space localizable in the ℓ^p norm, then it holds that $\|\tau_k \theta \cdot \Delta_j u\|_{\ell^p(L^p)} \sim \|\Delta_j u\|_p$. Hence,

$$\left(\sum_{k \in \mathbb{Z}^n} \|\tau_k \theta \cdot u\|_{B_{p,q}^s}^r \right)^{\frac{1}{r}} \leq c \left(\sum_{j=0}^{\infty} 2^{sjq} \|\Delta_j u\|_p^q \right)^{\frac{1}{q}} \leq c \|u\|_{B_{p,q}^s}.$$

Thus, $B_{p,q}^s \hookrightarrow (B_{p,q}^s)_{\ell^r}$.

(ii) Now, we will show that

$$\|u\|_{B_{p,q}^s} \leq c \|u\|_{(B_{p,q}^s)_{\ell^r}} \quad \text{for } c > 0.$$

Let $u \in \mathcal{S}'(\mathbb{R}^n)$. Then it holds that

$$\|L_j(u)\|_p = \|L_j(\sum_{k \in \mathbb{Z}^n} \tau_k \varphi \cdot u)\|_p = \|\sum_{k \in \mathbb{Z}^n} L_j(\tau_k \varphi \cdot u)\|_p \leq c(\sum_{k \in \mathbb{Z}^n} \|L_j(\tau_k \varphi \cdot u)\|_p^p)^{\frac{1}{p}}.$$

Since $r \leq \min(p, q)$, it holds that $\ell^r \hookrightarrow \ell^p$, i.e.

$$\|(\|L_j(\tau_k \varphi \cdot u)\|_p)_{k \in \mathbb{Z}^n}\|_{\ell^p} \leq c\|(\|L_j(\tau_k \varphi \cdot u)\|_p)_{k \in \mathbb{Z}^n}\|_{\ell^r}.$$

So, we have

$$\|L_j(u)\|_p \leq c(\sum_{k \in \mathbb{Z}^n} \|L_j(\tau_k \varphi \cdot u)\|_p^p)^{\frac{1}{p}} \leq c(\sum_{k \in \mathbb{Z}^n} \|L_j(\tau_k \varphi \cdot u)\|_p^r)^{\frac{1}{r}}.$$

This implies that

$$(\sum_{j=0}^{\infty} 2^{sjq} \|L_j u\|_p^q)^{\frac{1}{q}} \leq c(\sum_{j=0}^{\infty} 2^{sjq} (\sum_{k \in \mathbb{Z}^n} \|L_j(\tau_k \varphi \cdot u)\|_p^r)^{\frac{q}{r}})^{\frac{1}{q}}.$$

Consequently

$$(\sum_{j=0}^{\infty} 2^{sjq} \|L_j u\|_p^q)^{\frac{1}{q}} \leq c(\|(2^{sj} \|L_j(\tau_k \varphi \cdot u)\|_p)_{k \in \mathbb{Z}^n}\|_{\ell^q(\ell^r)}). \quad (6)$$

Since $r \leq \min(p, q)$, it holds that $r \leq q$. Then from Minkowski inequality we have

$$\|(2^{sj} \|L_j(\tau_k \varphi \cdot u)\|_p)_{k \in \mathbb{Z}^n}\|_{\ell^q(\ell^r)} \leq c\|(2^{sj} \|L_j(\tau_k \varphi \cdot u)\|_p)_{k \in \mathbb{Z}^n}\|_{\ell^r(\ell^q)}.$$

So, we can see that the inequality (6) becomes as follows

$$(\sum_{j=0}^{\infty} 2^{sjq} \|L_j u\|_p^q)^{\frac{1}{q}} \leq c(\|(2^{sj} \|L_j(\tau_k \varphi \cdot u)\|_p)_{k \in \mathbb{Z}^n}\|_{\ell^r(\ell^q)}).$$

Consequently

$$(\sum_{j=0}^{\infty} 2^{sjq} \|L_j u\|_p^q)^{\frac{1}{q}} \leq c(\sum_{k \in \mathbb{Z}^n} (\sum_{j=0}^{\infty} 2^{sjq} \|L_j(\tau_k \varphi \cdot u)\|_p^q)^{\frac{1}{q}})^{\frac{1}{r}} \leq c(\sum_{k \in \mathbb{Z}^n} \|\tau_k \varphi \cdot u\|_{B_{p,q}^s}^r)^{\frac{1}{r}} \leq c\|u\|_{(B_{p,q}^s)_{\ell^r}}.$$

Thus, $(B_{p,q}^s)_{\ell^r} \hookrightarrow B_{p,q}^s$. □

Remark 2. The generalization of Bourdaud Theorem given by Sickel and Smirnov in 1999 [9] using the wavelet method, the aim of this work is to generalize the same Theorem of a localization property by using a different method.

Theorem 3. Let $p \in [1, +\infty)$, $q \in [1, +\infty]$, $s \in \mathbb{R}$, $F_{p,q}^s$ and $(F_{p,q}^s)_{\ell^p}$ are respectively the Lizorkin-Triebel and localized Lizorkin-Triebel spaces. Then the space $F_{p,q}^s$ is localizable in the ℓ^p norm, i.e. $F_{p,q}^s = (F_{p,q}^s)_{\ell^p}$.

Proof. (i) $(F_{p,q}^s)_{\ell^p} \hookrightarrow F_{p,q}^s$. We will show that

$$\|f\|_{F_{p,q}^s} \leq c\|f\|_{(F_{p,q}^s)_{\ell^p}} \quad \text{for } c > 0.$$

From Definition 3, $\|f\|_{F_{p,q}^s} = \|(\sum_{j=0}^{\infty} 2^{sjq} |\Delta_j f|^q)^{\frac{1}{q}}\|_p$. We put $\Delta_j f = \sum_{k \in \mathbb{Z}^n} \tau_k \varphi \cdot \Delta_j f$, it follows that $\|f\|_{F_{p,q}^s} = \|(\sum_{j=0}^{\infty} (\sum_{k \in \mathbb{Z}^n} 2^{sjq} |\tau_k \varphi \Delta_j f|^q)^{\frac{1}{q}})\|_p$. Consequently

$$\|f\|_{F_{p,q}^s} = \| \|2^{sj} (\tau_k \varphi \Delta_j f)_{k \in \mathbb{Z}^n} \|_{\ell^q(\ell^1)} \|_p.$$

Since, $1 \leq q$. Then from Minkowski inequality we have

$$\|f\|_{F_{p,q}^s} = \| \|2^{sj} (\tau_k \varphi \Delta_j f)_{k \in \mathbb{Z}^n} \|_{\ell^q(\ell^1)} \|_p \leq \| \|2^{sj} (\tau_k \varphi \Delta_j f)_{k \in \mathbb{Z}^n} \|_{\ell^1(\ell^q)} \|_p.$$

Consequently

$$\|f\|_{F_{p,q}^s} \leq c \| \sum_{k \in \mathbb{Z}^n} (\sum_{j=0}^{\infty} 2^{sjq} |\tau_k \varphi \Delta_j f|^q)^{\frac{1}{q}} \|_p \leq c (\sum_{k \in \mathbb{Z}^n} \| (\sum_{j=0}^{\infty} 2^{sjq} |\tau_k \varphi \Delta_j f|^q)^{\frac{1}{q}} \|_p^p)^{\frac{1}{p}}.$$

Hence, $\|f\|_{F_{p,q}^s} \leq c (\sum_{k \in \mathbb{Z}^n} \|\tau_k \varphi \cdot f\|_{F_{p,q}^s}^p)^{\frac{1}{p}}$. Thus, $(F_{p,q}^s)_{\ell^p} \hookrightarrow F_{p,q}^s$.

(ii) $F_{p,q}^s \hookrightarrow (F_{p,q}^s)_{\ell^p}$. Now, we will show that

$$\|f\|_{(F_{p,q}^s)_{\ell^p}} \leq c \|f\|_{F_{p,q}^s} \quad \text{for } c > 0.$$

Let $p, q \in [1, +\infty]$ and $s \in \mathbb{R}$. Then it holds that

$$(\sum_{k \in \mathbb{Z}^n} \|\tau_k \varphi \cdot f\|_{F_{p,q}^s}^p)^{\frac{1}{p}} = (\sum_{k \in \mathbb{Z}^n} \|\tau_k \varphi \sum_{j=0}^{\infty} \Delta_j f\|_{F_{p,q}^s}^p)^{\frac{1}{p}} = (\sum_{k \in \mathbb{Z}^n} \| \sum_{j=0}^{\infty} \Delta_j f \tau_k \varphi \|_{F_{p,q}^s}^p)^{\frac{1}{p}}.$$

From Proposition 1, it follows that

$$\begin{aligned} (\sum_{k \in \mathbb{Z}^n} \|\tau_k \varphi \cdot f\|_{F_{p,q}^s}^p)^{\frac{1}{p}} &\leq c (\sum_{k \in \mathbb{Z}^n} \| (\sum_{j=0}^{\infty} 2^{sjq} |\Delta_j f \tau_k \varphi|^q)^{\frac{1}{q}} \|_p^p)^{\frac{1}{p}} \\ &\leq c (\sum_{k \in \mathbb{Z}^n} \|\tau_k \varphi (\sum_{j=0}^{\infty} 2^{sjq} |\Delta_j f|^q)^{\frac{1}{q}} \|_p^p)^{\frac{1}{p}}. \end{aligned}$$

Since L^p is a space localizable in the ℓ^p norm, then it holds that

$$(\sum_{k \in \mathbb{Z}^n} \|\tau_k \varphi \cdot f\|_{F_{p,q}^s}^p)^{\frac{1}{p}} \leq c \| (\sum_{j=0}^{\infty} 2^{sjq} |\Delta_j f|^q)^{\frac{1}{q}} \|_p \leq c \|f\|_{F_{p,q}^s}.$$

Thus, $F_{p,q}^s \hookrightarrow (F_{p,q}^s)_{\ell^p}$. □

3 CONCLUSION

In this work, we have generalized the Bourdaud theorem of a localization property of Besov spaces $B_{p,q}^s(\mathbb{R}^n)$ on the ℓ^r space, where $s \in \mathbb{R}$, $p, q, r \in [1, +\infty]$. Also, we have provided that the Lizorkin-Triebel spaces are localizable in the ℓ^p norm. In future work, we will investigate the localization property on other functional spaces.

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Поняття локалізаційної властивості просторів Бесова введено Г. Бурдо, він показав, що простори Бесова $B_{p,q}^s(\mathbb{R}^n)$, де $s \in \mathbb{R}$ і $p, q \in [1, +\infty]$ такі, що $p \neq q$, є нелокалізованими у нормі ℓ^p . Також він показав, що простори Бесова $B_{p,q}^s$ вкладені в локалізовані простори Бесова $(B_{p,q}^s)_{\ell^p}$ (тобто $B_{p,q}^s \hookrightarrow (B_{p,q}^s)_{\ell^p}$ при $p \geq q$). Також було показано, що локалізовані простори Бесова $(B_{p,q}^s)_{\ell^p}$ вкладені в простори Бесова $B_{p,q}^s$ (тобто $(B_{p,q}^s)_{\ell^p} \hookrightarrow B_{p,q}^s$ при $p \leq q$). Зокрема $B_{p,p}^s$ є локалізованим в нормі ℓ^p , де ℓ^p простір послідовностей $(a_k)_k$ таких, що $\|(a_k)\|_{\ell^p} < \infty$. У цій статті ми узагальнили теорему Бурдо про локалізаційну властивість просторів Бесова $B_{p,q}^s(\mathbb{R}^n)$ на простір ℓ^r , де $r \in [1, +\infty]$. А точніше ми довели, що будь-який простір Бесова $B_{p,q}^s$ є вкладений в локалізований простір Бесова $(B_{p,q}^s)_{\ell^r}$ (тобто $B_{p,q}^s \hookrightarrow (B_{p,q}^s)_{\ell^r}$ при $r \geq \max(p, q)$). Також ми показали, що будь-який локалізований простір Бесова $(B_{p,q}^s)_{\ell^r}$ вкладений в простір Бесова $B_{p,q}^s$ (тобто $(B_{p,q}^s)_{\ell^r} \hookrightarrow B_{p,q}^s$ при $r \leq \min(p, q)$). І на завершення було показано, що простори Лізоркіна-Трібеля $F_{p,q}^s(\mathbb{R}^n)$, де $s \in \mathbb{R}$ і $p, q \in [1, +\infty]$ є локалізованими в нормі ℓ^p (тобто $F_{p,q}^s = (F_{p,q}^s)_{\ell^p}$).

Ключові слова і фрази: простори Бесова, простори Лізоркіна-Трібеля, локалізаційна властивість.



FILEVYCH P.V., HRYBEL O.B.

THE GROWTH OF THE MAXIMAL TERM OF DIRICHLET SERIES

Let Λ be the class of nonnegative sequences (λ_n) increasing to $+\infty$, $A \in (-\infty, +\infty]$, L_A be the class of continuous functions increasing to $+\infty$ on a half-closed interval of the form $[A_0, A)$, and $F(s) = \sum a_n e^{s\lambda_n}$ be a Dirichlet series such that its maximum term $\mu(\sigma, F) = \max_n |a_n| e^{\sigma\lambda_n}$ is defined for every $\sigma \in (-\infty, A)$. It is proved that for all functions $\alpha \in L_{+\infty}$ and $\beta \in L_A$ the equality

$$\rho_{\alpha, \beta}^*(F) = \max_{(\eta_n) \in \Lambda} \overline{\lim}_{n \rightarrow \infty} \frac{\alpha(\eta_n)}{\beta\left(\frac{\eta_n}{\lambda_n} + \frac{1}{\lambda_n} \ln \frac{1}{|a_n|}\right)}$$

holds, where $\rho_{\alpha, \beta}^*(F)$ is the generalized α, β -order of the function $\ln \mu(\sigma, F)$, i.e. $\rho_{\alpha, \beta}^*(F) = 0$ if the function $\mu(\sigma, F)$ is bounded on $(-\infty, A)$, and $\rho_{\alpha, \beta}^*(F) = \overline{\lim}_{\sigma \uparrow A} \alpha(\ln \mu(\sigma, F)) / \beta(\sigma)$ if the function $\mu(\sigma, F)$ is unbounded on $(-\infty, A)$.

Key words and phrases: Dirichlet series, maximal term, central index, generalized order.

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1 INTRODUCTION

We denote by \mathbb{N}_0 the class of nonnegative integer numbers, and let Λ be the class of nonnegative increasing sequences $\lambda = (\lambda_n)_{n \in \mathbb{N}_0}$ tending to $+\infty$.

Let $\lambda \in \Lambda$. We consider a Dirichlet series of the form

$$F(s) = \sum_{n=0}^{\infty} a_n e^{s\lambda_n}, \quad s = \sigma + it, \quad (1)$$

and set

$$\sigma^*(F) = \overline{\lim}_{n \rightarrow \infty} \frac{1}{\lambda_n} \ln \frac{1}{|a_n|}, \quad E(F) = \{\sigma \in \mathbb{R} : |a_n| e^{\sigma\lambda_n} = o(1), n \rightarrow \infty\}.$$

It is easy to see that

$$\sigma^*(F) = \begin{cases} -\infty, & \text{if } E(F) = \emptyset; \\ \sup E(F), & \text{if } E(F) \neq \emptyset. \end{cases}$$

If $\sigma^*(F) > -\infty$, then for all $\sigma \in (-\infty; \sigma^*(F))$ we define the maximal term and central index of the series F respectively by

$$\mu(\sigma, F) = \max\{|a_n| e^{\sigma\lambda_n} : n \in \mathbb{N}_0\}, \quad \nu(\sigma, F) = \max\{n \in \mathbb{N}_0 : |a_n| e^{\sigma\lambda_n} = \mu(\sigma, F)\}.$$

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Let $A \in (-\infty, +\infty]$, and $\alpha : D_\alpha \rightarrow \mathbb{R}$ be a real function. We say that $\alpha \in L_A$ if two following conditions are fulfilled: (i) the domain D_α of α is a half-closed interval of the form $[A_0, A)$; (ii) the function α is continuous and increasing to $+\infty$ on D_α . If $\alpha \in L_A$ and $A \leq x \leq +\infty$, then we assume that $\alpha(x) = +\infty$.

For a given $A \in (-\infty, +\infty]$ and $\lambda \in \Lambda$ we denote by $\mathcal{D}_A^*(\lambda)$ the class of Dirichlet series of the form (1) such that $\sigma^*(F) \geq A$ and put $\mathcal{D}_A^* = \cup_{\lambda \in \Lambda} \mathcal{D}_A^*(\lambda)$.

Let $\alpha \in L_{+\infty}$, $\beta \in L_A$ and $F \in \mathcal{D}_A^*$. If the function $\mu(\sigma, F)$ is bounded on $(-\infty, A)$, we set $\rho_{\alpha, \beta}^*(F) = 0$; if the function $\mu(\sigma, F)$ is unbounded on $(-\infty, A)$, we put

$$\rho_{\alpha, \beta}^*(F) = \overline{\lim}_{\sigma \uparrow A} \frac{\alpha(\ln \mu(\sigma, F))}{\beta(\sigma)}.$$

Let p be a positive constant. Under some conditions on functions $\alpha, \beta \in L_{+\infty}$, Sheremeta [1] proved that

$$\rho_{\alpha, \beta}^*(F) = \overline{\lim}_{n \rightarrow \infty} \frac{\alpha(\lambda_n/p)}{\beta\left(\frac{1}{p} + \frac{1}{\lambda_n} \ln \frac{1}{|a_n|}\right)} \quad (2)$$

for every Dirichlet series $F \in \mathcal{D}_{+\infty}^*$ of the form (1). Note that without additional conditions on functions $\alpha, \beta \in L_{+\infty}$ formula (1) is false in general (see e.g. [2, 3]).

The following theorem indicates a formula for calculating $\rho_{\alpha, \beta}^*(F)$ in the case of arbitrary $A \in (-\infty, +\infty]$, $\alpha \in L_{+\infty}$, $\beta \in L_A$, and $F \in \mathcal{D}_A^*$.

Theorem 1. *Let $A \in (-\infty, +\infty]$, $\alpha \in L_{+\infty}$, $\beta \in L_A$. Then for every Dirichlet series $F \in \mathcal{D}_A^*$ of the form (1) we have*

$$\rho_{\alpha, \beta}^*(F) = \max_{\eta \in \Lambda} \overline{\lim}_{n \rightarrow \infty} \frac{\alpha(\eta_n)}{\beta\left(\frac{\eta_n}{\lambda_n} + \frac{1}{\lambda_n} \ln \frac{1}{|a_n|}\right)}.$$

It can easily be shown that Theorem 1 is equivalent to the following theorem.

Theorem 2. *Let $A \in (-\infty, +\infty]$, $\alpha \in L_{+\infty}$, $\beta \in L_A$. Then for every Dirichlet series $F \in \mathcal{D}_A^*$ of the form (1) we have*

$$\rho_{\alpha, \beta}^*(F) = \overline{\lim}_{n \rightarrow \infty} \sup_{x \in D_\alpha} \frac{\alpha(x)}{\beta\left(\frac{x}{\lambda_n} + \frac{1}{\lambda_n} \ln \frac{1}{|a_n|}\right)}.$$

2 PROOF OF THEOREM 1

For a sequence $\eta \in \Lambda$ set $k(\eta) = \overline{\lim}_{n \rightarrow \infty} \frac{\alpha(\eta_n)}{\beta\left(\frac{\eta_n}{\lambda_n} + \frac{1}{\lambda_n} \ln \frac{1}{|a_n|}\right)}$.

Consider a Dirichlet series $F \in \mathcal{D}^*$ of the form (1) and prove that $k(\eta) \leq \rho_{\alpha, \beta}^*(F)$. If $\rho_{\alpha, \beta}^*(F) = +\infty$ it is trivial. Assume that $\rho_{\alpha, \beta}^*(F) < +\infty$, and let $\rho > \rho_{\alpha, \beta}^*(F)$ be a constant. Then

$$\ln \mu(\sigma) \leq \alpha^{-1}(\rho \beta(\sigma)), \quad \sigma \in [\sigma_0, A).$$

Hence, for every $n \in \mathbb{N}_0$ we have $\ln |a_n| \leq \alpha^{-1}(\rho \beta(\sigma)) - \lambda_n \sigma$, $\sigma \in [\sigma_0, A)$. Therefore, using the notation $\sigma_n = \beta^{-1}\left(\frac{1}{\rho} \alpha(\eta_n)\right)$ for all $n \geq n_0$ we obtain

$$\ln |a_n| \leq \alpha^{-1}(\rho \beta(\sigma_n)) - \lambda_n \sigma_n = \eta_n - \lambda_n \beta^{-1}\left(\frac{1}{\rho} \alpha(\eta_n)\right),$$

and this can also be written as

$$\rho \geq \frac{\alpha(\eta_n)}{\beta\left(\frac{\eta_n}{\lambda_n} + \frac{1}{\lambda_n} \ln \frac{1}{|a_n|}\right)}, \quad n \geq n_0.$$

This yields the inequality $k(\eta) \leq \rho$. Since $\rho > \rho_{\alpha,\beta}^*(F)$ is an arbitrary, we obtain $k(\eta) \leq \rho_{\alpha,\beta}^*(F)$.

It remains to show that there exists a sequence $\eta \in \Lambda$ such that $k(\eta) \geq \rho_{\alpha,\beta}^*(F)$. If $\rho_{\alpha,\beta}^*(F) = 0$, then $k(\eta) \geq \rho_{\alpha,\beta}^*(F)$ for every sequence $\eta \in \Lambda$ as is proved above. Let $\rho_{\alpha,\beta}^*(F) > 0$, and $(\rho_k)_{k \in \mathbb{N}_0}$ be a positive sequence that increase to $\rho_{\alpha,\beta}^*(F)$. Then it follows from the definition of $\rho_{\alpha,\beta}^*(F)$ that there exists a sequence $(\sigma_k)_{k \in \mathbb{N}_0}$ increasing to A such that the sequence $(\nu(\sigma_k, F))_{k \in \mathbb{N}_0}$ is also increasing and

$$\alpha(\ln \mu(\sigma_k)) \geq \rho_k \beta(\sigma_k), \quad k \in \mathbb{N}_0.$$

Let $n_k = \nu(\sigma_k, F)$, $k \in \mathbb{N}_0$. Consider a sequence $\eta \in \Lambda$ such that $\eta_{n_k} = \alpha^{-1}(\rho_k \beta(\sigma_k))$, $k \geq k_0$. Then for every $k \geq k_0$ we have

$$\ln |a_{n_k}| + \lambda_{n_k} \beta^{-1}\left(\frac{1}{\rho_k} \alpha(\eta_{n_k})\right) = \ln |a_{n_k}| + \lambda_{n_k} \sigma_k = \ln \mu(\sigma_k) \geq \alpha^{-1}(\rho_k \beta(\sigma_k)) = \eta_{n_k}.$$

This yields $\rho_k < \frac{\alpha(\eta_{n_k})}{\beta\left(\frac{\eta_{n_k}}{\lambda_{n_k}} + \frac{1}{\lambda_{n_k}} \ln \frac{1}{|a_{n_k}|}\right)}$ for all sufficiently large k .

Therefore, $k(\eta) \geq \overline{\lim}_{k \rightarrow \infty} \rho_k = \rho_{\alpha,\beta}^*(F)$. Theorem 1 is proved.

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Нехай Λ — клас невід'ємних зростаючих до $+\infty$ послідовностей (λ_n) , $A \in (-\infty, +\infty]$, L_A — клас неперервних зростаючих до $+\infty$ функцій, заданих на напіввідкритому інтервалі вигляду $[A_0, A)$, а $F(s) = \sum a_n e^{s\lambda_n}$ — ряд Діріхле такий, що його максимальний член $\mu(\sigma, F) = \max_n |a_n| e^{\sigma\lambda_n}$ є визначеним для всіх $\sigma \in (-\infty, A)$. В роботі доведено, що для довільних функцій $\alpha \in L_{+\infty}$ і $\beta \in L_A$ правильна рівність

$$\rho_{\alpha,\beta}^*(F) = \max_{(\eta_n) \in \Lambda} \overline{\lim}_{n \rightarrow \infty} \frac{\alpha(\eta_n)}{\beta\left(\frac{\eta_n}{\lambda_n} + \frac{1}{\lambda_n} \ln \frac{1}{|a_n|}\right)},$$

де $\rho_{\alpha,\beta}^*(F)$ — узагальнений α, β -порядок функції $\ln \mu(\sigma, F)$, тобто $\rho_{\alpha,\beta}^*(F) = 0$, якщо функція $\mu(\sigma, F)$ обмежена на $(-\infty, A)$, і $\rho_{\alpha,\beta}^*(F) = \overline{\lim}_{\sigma \uparrow A} \alpha(\ln \mu(\sigma, F)) / \beta(\sigma)$, якщо функція $\mu(\sigma, F)$ необмежена на $(-\infty, A)$.

Ключові слова і фрази: ряд Діріхле, максимальний член, центральний індекс, узагальнений порядок.



FREI M.M.

WICK CALCULUS ON SPACES OF REGULAR GENERALIZED FUNCTIONS OF LÉVY WHITE NOISE ANALYSIS

Many objects of the Gaussian white noise analysis (spaces of test and generalized functions, stochastic integrals and derivatives, etc.) can be constructed and studied in terms of so-called chaotic decompositions, based on a *chaotic representation property* (CRP): roughly speaking, any square integrable with respect to the Gaussian measure random variable can be decomposed in a series of Itô's stochastic integrals from nonrandom functions. In the Lévy analysis there is no the CRP (except the Gaussian and Poissonian particular cases). Nevertheless, there are different generalizations of this property. Using these generalizations, one can construct different spaces of test and generalized functions. And in any case it is necessary to introduce a natural product on spaces of generalized functions, and to study related topics. This product is called a *Wick product*, as in the Gaussian analysis.

The construction of the Wick product in the Lévy analysis depends, in particular, on the selected generalization of the CRP. In this paper we deal with Lytvynov's generalization of the CRP and with the corresponding spaces of regular generalized functions. The goal of the paper is to introduce and to study the Wick product on these spaces, and to consider some related topics (Wick versions of holomorphic functions, interconnection of the Wick calculus with operators of stochastic differentiation). Main results of the paper consist in study of properties of the Wick product and of the Wick versions of holomorphic functions. In particular, we proved that an operator of stochastic differentiation is a differentiation (satisfies the Leibniz rule) with respect to the Wick multiplication.

Key words and phrases: Lévy process, stochastic differentiation, Wick product.

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INTRODUCTION

Due to development of physics and mathematics (in particular, of the quantum field theory, of the mathematical physics, of the theory of random processes) there is a need to develop a theory of test and generalized functions of infinitely many variables. There are different approaches to building of such a theory. Correspondingly, different spaces of test and generalized functions are the object of study. One of the most successful approaches consists in building of the just now mentioned spaces in such a way that the natural pairing between test and generalized functions is generated by integration with respect to some probability measure on a dual nuclear space (in particular, on a dual Schwartz space). First it was the standard Gaussian measure (the measure of a Gaussian white noise), the corresponding theory is called the *Gaussian white noise analysis* (see, e.g., [10, 21]); then it were realized numerous generalizations. In particular, important for applications results can be obtained if as the above-mentioned

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measure one uses a so-called *Lévy white noise measure* (e.g., [4, 5]), the corresponding theory is called the *Lévy white noise analysis*.

An important role in the Gaussian analysis belongs to a so-called *chaotic representation property* (CRP): roughly speaking, any square integrable with respect to the Gaussian measure random variable can be decomposed in a series of Itô's stochastic integrals from nonrandom functions. In particular, the CRP can be used in order to construct the extended Skorohod stochastic integral [13, 27] and the Hida stochastic derivative [10].

Unfortunately, in the Lévy analysis there is no the CRP [29] (except Gaussian and Poissonian particular cases). Nevertheless, there are different approaches to a generalization of this property: Itô's approach [12], Nualart-Schoutens' approach [24, 25], Lytvynov's approach [23], Oksendal's approach [4, 5], etc. The interconnections between these generalizations of the CRP are described in, in particular, [1, 4, 5, 17, 23, 28, 30].

One can use different generalizations of the CRP and construct different spaces of test and generalized functions in the Lévy analysis, depending on the purpose of the research. And in any case, for solving of some problems, or even simply for the completeness of the theory, it is necessary to introduce a natural product on spaces of generalized functions, and to study related topics. In the classical Gaussian analysis such a product, known as a *Wick product*, can be introduced with use of symmetric tensor products of kernels from natural decompositions of generalized functions (e.g., [22]). But in a general Lévy analysis the situation is more complicated: now the construction of a product on spaces of generalized functions appreciably depends on the construction of the just now mentioned spaces that, by-turn, depends, in particular, on the selected generalization of the CRP. For example, elements of the Lévy analysis in terms of Oksendal's generalization of the CRP and, in particular, the corresponding Wick product and related topics, are considered in [4, 5].

In this paper we deal with so-called regular parametrized Kondratiev-type spaces of generalized functions of the Lévy white noise analysis [16], which are constructed with use of Lytvynov's generalization of the CRP. The goal of the paper is to introduce and to study a natural product (a Wick product) on these spaces, and to consider some related topics (Wick versions of holomorphic functions, stochastic equations with Wick type nonlinearities, interconnection of the Wick calculus with operators of stochastic differentiation). Main results of the paper consist in study of properties of the Wick product and of the Wick versions of holomorphic functions. In particular, we proved that an operator of stochastic differentiation is a differentiation (satisfies the Leibniz rule) with respect to the Wick multiplication.

Note that, as distinguished from the Gaussian case, now the symmetric tensor product of kernels from natural decompositions of generalized functions is indeterminated, therefore we introduce an applicable generalization of this product, by analogy with a so-called Gamma white noise analysis [15] and a more general Meixner white noise analysis [14].

The paper is organized in the following manner. In the first section we recall necessary notions, definitions and statements. Namely, we introduce a Lévy process L and convenient for our considerations probability space connected with L ; describe in detail Lytvynov's generalization of the CRP; consider a regular parametrized rigging of (L^2) , and the stochastic integrals, derivatives, and operators of stochastic differentiation on the spaces that belong to this rigging. The second section is devoted to the Wick calculus: in the first subsection we introduce and study the Wick product and the Wick versions of holomorphic functions on the spaces of regular generalized functions; in the second subsection we study an interconnection

between the Wick calculus and the operator of stochastic differentiation.

1 PRELIMINARIES

In this paper we accept on default that signs $\|\cdot\|_H$ or $|\cdot|_H$ denote the norm in a space H ; a sign $(\cdot, \cdot)_H$ denotes the scalar product in H ; signs $\langle \cdot, \cdot \rangle_H$ or $\langle\langle \cdot, \cdot \rangle\rangle_H$ denote the dual pairing generated by the scalar product in H .

1.1 Lévy processes

Set $\mathbb{R}_+ := [0, +\infty)$. Consider a real-valued locally square integrable Lévy process $L = (L_t)_{t \in \mathbb{R}_+}$ (i.e. a random process on \mathbb{R}_+ with stationary independent increments and such that $L_0 = 0$) without Gaussian part and drift. As is known (e.g., [5]), the characteristic function of L is

$$\mathbb{E}[e^{i\theta L_t}] = \exp \left[t \int_{\mathbb{R}} (e^{i\theta x} - 1 - i\theta x) \nu(dx) \right]. \quad (1)$$

Here ν is the Lévy measure of L , which is a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, here and below \mathcal{B} denotes the Borel σ -algebra; \mathbb{E} denotes the expectation. We assume that ν is a Radon measure whose support contains an infinite number of points, $\nu(\{0\}) = 0$, there exists $\varepsilon > 0$ such that

$$\int_{\mathbb{R}} x^2 e^{\varepsilon|x|} \nu(dx) < \infty,$$

and

$$\int_{\mathbb{R}} x^2 \nu(dx) = 1. \quad (2)$$

Let us define the measure of the white noise of L . By \mathcal{D} denote the set of all real-valued infinite-differentiable functions on \mathbb{R}_+ with compact supports. As is known, \mathcal{D} can be endowed by the projective limit topology generated by a family of Sobolev spaces (e.g., [3]). Let \mathcal{D}' be the set of linear continuous functionals on \mathcal{D} . For $\omega \in \mathcal{D}'$ and $\varphi \in \mathcal{D}$ denote $\omega(\varphi)$ by $\langle \omega, \varphi \rangle$; note that one can understand $\langle \cdot, \cdot \rangle$ as the dual pairing generated by the scalar product in the space $L^2(\mathbb{R}_+)$ of (classes of) square integrable with respect to the Lebesgue measure real-valued functions on \mathbb{R}_+ (e.g., [3]). The notation $\langle \cdot, \cdot \rangle$ will be preserved for dual pairings in tensor powers of riggings of $L^2(\mathbb{R}_+)$ and in tensor powers of complexifications of such riggings.

Definition 1. A probability measure μ on $(\mathcal{D}', \mathcal{C}(\mathcal{D}'))$, where \mathcal{C} denotes the cylindrical σ -algebra, with the Fourier transform

$$\int_{\mathcal{D}'} e^{i\langle \omega, \varphi \rangle} \mu(d\omega) = \exp \left[\int_{\mathbb{R}_+ \times \mathbb{R}} (e^{i\varphi(u)x} - 1 - i\varphi(u)x) d\nu(dx) \right], \quad \varphi \in \mathcal{D}, \quad (3)$$

is called the measure of a Lévy white noise.

The existence of μ follows from the Bochner-Minlos theorem (e.g., [11]), this proved in [23]. Below we assume that the σ -algebra $\mathcal{C}(\mathcal{D}')$ is completed with respect to μ , i.e. we take the completion of $\mathcal{C}(\mathcal{D}')$ and preserve for this completion the previous designation. So, now $\mathcal{C}(\mathcal{D}')$ contains all subsets of all measurable sets O such that $\mu(O) = 0$.

Denote by $(L^2) := L^2(\mathcal{D}', \mathcal{C}(\mathcal{D}'), \mu)$ the space of (classes of) complex-valued square integrable with respect to μ functions on \mathcal{D}' . Let $f \in L^2(\mathbb{R}_+)$ and a sequence $(\varphi_k \in \mathcal{D})_{k \in \mathbb{N}}$

converge to f in $L^2(\mathbb{R}_+)$ as $k \rightarrow \infty$ (as is well known (e.g., [3]), \mathcal{D} is a dense set in $L^2(\mathbb{R}_+)$). One can show [4, 5, 17, 23] that $\langle \circ, f \rangle := (L^2) - \lim_{k \rightarrow \infty} \langle \circ, \varphi_k \rangle$ (i.e. the limit in the topology of the space (L^2)) is well-defined as an element of (L^2) .

Denote by 1_A the indicator of a set A . Set $1_{[0,0]} \equiv 0$ and consider $\langle \circ, 1_{[0,t]} \rangle \in (L^2)$, $t \in \mathbb{R}_+$. It follows from (1) and (3) that $(\langle \circ, 1_{[0,t]} \rangle)_{t \in \mathbb{R}_+}$ can be identified with a Lévy process on the probability space $(\mathcal{D}', \mathcal{C}(\mathcal{D}'), \mu)$ (see, e.g., [4, 5]). So, one can write $L_t = \langle \circ, 1_{[0,t]} \rangle \in (L^2)$.

1.2 Lytvynov's generalization of the CRP

Denote by $\hat{\otimes}$ a symmetric tensor product, by a subscript \mathbb{C} —complexifications of spaces. Set $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. Denote by \mathcal{P} the set of complex-valued polynomials on \mathcal{D}' that consists of zero and elements of the form

$$f(\omega) = \sum_{n=0}^{N_f} \langle \omega^{\otimes n}, f^{(n)} \rangle, \quad \omega \in \mathcal{D}', \quad N_f \in \mathbb{Z}_+, \quad f^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\hat{\otimes} n}, \quad f^{(N_f)} \neq 0,$$

here N_f is called the *power of a polynomial* f ; $\langle \omega^{\otimes 0}, f^{(0)} \rangle := f^{(0)} \in \mathcal{D}_{\mathbb{C}}^{\hat{\otimes} 0} := \mathbb{C}$. The measure μ of a Lévy white noise has a holomorphic at zero Laplace transform (this follows from (3) and properties of the measure ν , see also [23]), therefore \mathcal{P} is a dense set in (L^2) [26]. Denote by \mathcal{P}_n the set of polynomials of power smaller or equal to n , by $\overline{\mathcal{P}}_n$ the closure of \mathcal{P}_n in (L^2) . Let for $n \in \mathbb{N}$ $\mathbf{P}_n := \overline{\mathcal{P}}_n \ominus \overline{\mathcal{P}}_{n-1}$ (the orthogonal difference in (L^2)), $\mathbf{P}_0 := \overline{\mathcal{P}}_0$. It is clear that

$$(L^2) = \bigoplus_{n=0}^{\infty} \mathbf{P}_n.$$

Let $f^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\hat{\otimes} n}$, $n \in \mathbb{Z}_+$. Denote by $:\langle \circ^{\otimes n}, f^{(n)} \rangle:$ the orthogonal projection of a monomial $\langle \circ^{\otimes n}, f^{(n)} \rangle$ onto \mathbf{P}_n . Let us define *real* (i.e. *bilinear*) scalar products $(\cdot, \cdot)_{ext}$ on $\mathcal{D}_{\mathbb{C}}^{\hat{\otimes} n}$, $n \in \mathbb{Z}_+$, by setting for $f^{(n)}, g^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\hat{\otimes} n}$

$$(f^{(n)}, g^{(n)})_{ext} := \frac{1}{n!} \int_{\mathcal{D}'} : \langle \omega^{\otimes n}, f^{(n)} \rangle :: \langle \omega^{\otimes n}, g^{(n)} \rangle : \mu(d\omega). \quad (4)$$

This definition is well posed: it is clear that $(\cdot, \cdot)_{ext}$ are *quasiscalar* products on $\mathcal{D}_{\mathbb{C}}^{\hat{\otimes} n}$, the fact that these products are *scalar* follows from their explicit formula calculated in [23] (see formula (6) below).

By $|\cdot|_{ext}$ we denote the norms corresponding to scalar products (4), i.e.

$$|f^{(n)}|_{ext} := \sqrt{(f^{(n)}, f^{(n)})_{ext}}$$

Denote by $\mathcal{H}_{ext}^{(n)}$, $n \in \mathbb{Z}_+$, the completions of $\mathcal{D}_{\mathbb{C}}^{\hat{\otimes} n}$ with respect to the norms $|\cdot|_{ext}$. For $F^{(n)} \in \mathcal{H}_{ext}^{(n)}$ define a Wick monomial $:\langle \circ^{\otimes n}, F^{(n)} \rangle:$ $\stackrel{\text{def}}{=} (L^2) - \lim_{k \rightarrow \infty} :\langle \circ^{\otimes n}, f_k^{(n)} \rangle:$, where $\mathcal{D}_{\mathbb{C}}^{\hat{\otimes} n} \ni f_k^{(n)} \rightarrow F^{(n)}$ as $k \rightarrow \infty$ in $\mathcal{H}_{ext}^{(n)}$ (the well-posedness of this definition can be proved by the method of "mixed sequences"). Since, as is easy to see, for each $n \in \mathbb{Z}_+$ the set $\{:\langle \circ^{\otimes n}, f^{(n)} \rangle: | f^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\hat{\otimes} n}\}$ is dense in \mathbf{P}_n , $F \in (L^2)$ if and only if there exists a unique sequence of kernels $F^{(n)} \in \mathcal{H}_{ext}^{(n)}$, $n \in \mathbb{Z}_+$, such that

$$F = \sum_{n=0}^{\infty} :\langle \circ^{\otimes n}, F^{(n)} \rangle: \quad (5)$$

(the series converges in (L^2)) and

$$\|F\|_{(L^2)}^2 = \int_{\mathcal{D}'} |F(\omega)|^2 \mu(d\omega) = \mathbb{E}|F|^2 = \sum_{n=0}^{\infty} n! |F^{(n)}|_{ext}^2 < \infty.$$

So, for $F, G \in (L^2)$ the *real* scalar product has a form

$$(F, G)_{(L^2)} = \int_{\mathcal{D}'} F(\omega)G(\omega)\mu(d\omega) = \mathbb{E}[FG] = \sum_{n=0}^{\infty} n!(F^{(n)}, G^{(n)})_{ext},$$

where $F^{(n)}, G^{(n)} \in \mathcal{H}_{ext}^{(n)}$ are the kernels from decompositions (5) for F and G respectively. In particular, for $F^{(n)} \in \mathcal{H}_{ext}^{(n)}$ and $G^{(m)} \in \mathcal{H}_{ext}^{(m)}$, $n, m \in \mathbb{Z}_+$,

$$\begin{aligned} (: \langle \circ^{\otimes n}, F^{(n)} \rangle :; : \langle \circ^{\otimes m}, G^{(m)} \rangle :)_{(L^2)} &= \int_{\mathcal{D}'} : \langle \omega^{\otimes n}, F^{(n)} \rangle :: \langle \omega^{\otimes m}, G^{(m)} \rangle : \mu(d\omega) \\ &= \mathbb{E}[: \langle \circ^{\otimes n}, F^{(n)} \rangle :: \langle \circ^{\otimes m}, G^{(m)} \rangle :] = \delta_{n,m} n! (F^{(n)}, G^{(n)})_{ext}. \end{aligned}$$

Also we note that in the space $(L^2) : \langle \circ^{\otimes 0}, F^{(0)} \rangle : = \langle \circ^{\otimes 0}, F^{(0)} \rangle = F^{(0)}$ and $: \langle \circ, F^{(1)} \rangle : = \langle \circ, F^{(1)} \rangle$ [23].

In what follows, we need an explicit formula for the scalar products $(\cdot, \cdot)_{ext}$. Let us write out this formula. Denote by $\|\cdot\|_v$ the norm in the space $L^2(\mathbb{R}, \nu)$ of (classes of) square integrable with respect to the Lévy measure ν (see (1)) real-valued functions on \mathbb{R} . Let

$$p_n(x) := x^n + a_{n,n-1}x^{n-1} + \dots + a_{n,1}x, \quad a_{n,j} \in \mathbb{R}, \quad j \in \{1, \dots, n-1\}, \quad n \in \mathbb{N},$$

be polynomials orthogonal in $L^2(\mathbb{R}, \nu)$, i.e. for natural numbers n, m such that $n \neq m$, $\int_{\mathbb{R}} p_n(x)p_m(x)\nu(dx) = 0$. Then, as it follows from [23], for $F^{(n)}, G^{(n)} \in \mathcal{H}_{ext}^{(n)}$, $n \in \mathbb{N}$,

$$\begin{aligned} (F^{(n)}, G^{(n)})_{ext} &\equiv (F^{(n)}, G^{(n)})_{\mathcal{H}_{ext}^{(n)}} \\ &= \sum_{\substack{k, l, j, s_j \in \mathbb{N}: j=1, \dots, k, l_1 > l_2 > \dots > l_k, \\ l_1 s_1 + \dots + l_k s_k = n}} \frac{n!}{s_1! \dots s_k!} \left(\frac{\|p_{l_1}\|_v}{l_1!} \right)^{2s_1} \dots \left(\frac{\|p_{l_k}\|_v}{l_k!} \right)^{2s_k} \\ &\times \int_{\mathbb{R}_+^{s_1 + \dots + s_k}} F^{(n)}(\underbrace{u_1, \dots, u_1}_{l_1}, \dots, \underbrace{u_{s_1}, \dots, u_{s_1}}_{l_1}, \dots, \underbrace{u_{s_1 + \dots + s_k}, \dots, u_{s_1 + \dots + s_k}}_{l_k}) \\ &\times G^{(n)}(\underbrace{u_1, \dots, u_1}_{l_1}, \dots, \underbrace{u_{s_1}, \dots, u_{s_1}}_{l_1}, \dots, \underbrace{u_{s_1 + \dots + s_k}, \dots, u_{s_1 + \dots + s_k}}_{l_k}) du_1 \dots du_{s_1 + \dots + s_k}. \end{aligned} \quad (6)$$

In particular, for $n = 1$

$$(F^{(1)}, G^{(1)})_{ext} \equiv (F^{(1)}, G^{(1)})_{\mathcal{H}_{ext}^{(1)}} = \|p_1\|_v^2 \int_{\mathbb{R}_+} F^{(1)}(u)G^{(1)}(u)du = (F^{(1)}, G^{(1)})_{L^2(\mathbb{R}_+)_\mathbb{C}} \quad (7)$$

(by (2) $\|p_1\|_v^2 = \int_{\mathbb{R}} x^2 \nu(dx) = 1$); in the case $n = 2$ we have

$$\begin{aligned} (F^{(2)}, G^{(2)})_{ext} &\equiv (F^{(2)}, G^{(2)})_{\mathcal{H}_{ext}^{(2)}} = \|p_1\|_v^4 \int_{\mathbb{R}_+^2} F^{(2)}(u_1, u_2)G^{(2)}(u_1, u_2)du_1 du_2 \\ &+ \frac{\|p_2\|_v^2}{2} \int_{\mathbb{R}_+} F^{(2)}(u, u)G^{(2)}(u, u)du = (F^{(2)}, G^{(2)})_{L^2(\mathbb{R}_+)_\mathbb{C}^{\otimes 2}} + \frac{\|p_2\|_v^2}{2} \int_{\mathbb{R}_+} F^{(2)}(u, u)G^{(2)}(u, u)du, \end{aligned}$$

etc.

Remark. Note that the explicit formula for scalar products in $\mathcal{H}_{ext}^{(n)}$, $n \in \mathbb{Z}_+$, calculated in [23], differs from (6). But it is very easy to verify that actually these formulas differ by the record form only.

Denote $\mathcal{H} := L^2(\mathbb{R}_+)$, then $\mathcal{H}_{\mathbb{C}} = L^2(\mathbb{R}_+)_{\mathbb{C}}$ (in what follows, this notation will be used very often). It follows from (7) that $\mathcal{H}_{ext}^{(1)} = \mathcal{H}_{\mathbb{C}}$; and, as is easily seen, for $n \in \mathbb{N} \setminus \{1\}$ one can identify $\mathcal{H}_{\mathbb{C}}^{\otimes n}$ with the proper subspace of $\mathcal{H}_{ext}^{(n)}$ that consists of "vanishing on diagonals" elements (roughly speaking, such that $F^{(n)}(u_1, \dots, u_n) = 0$ if there exist $k, j \in \{1, \dots, n\}$: $k \neq j$, but $u_k = u_j$). In this sense the space $\mathcal{H}_{ext}^{(n)}$ is an extension of $\mathcal{H}_{\mathbb{C}}^{\otimes n}$ (this explains why we use the subscript "ext" in our designations).

1.3 A regular rigging of (L^2)

Denote $\mathcal{P}_W := \{f = \sum_{n=0}^{N_f} \langle \circ^{\otimes n}, f^{(n)} \rangle, f^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\otimes n}, N_f \in \mathbb{Z}_+\} \subset (L^2)$. Accept on default $\beta \in [0, 1]$, $q \in \mathbb{Z}$ in the case $\beta \in (0, 1]$ and $q \in \mathbb{Z}_+$ if $\beta = 0$. Define real (bilinear) scalar products $(\cdot, \cdot)_{q, \beta}$ on \mathcal{P}_W by setting for

$$f = \sum_{n=0}^{N_f} \langle \circ^{\otimes n}, f^{(n)} \rangle, g = \sum_{n=0}^{N_g} \langle \circ^{\otimes n}, g^{(n)} \rangle \in \mathcal{P}_W$$

$$(f, g)_{q, \beta} := \sum_{n=0}^{\min(N_f, N_g)} (n!)^{1+\beta} 2^{qn} (f^{(n)}, g^{(n)})_{ext}.$$

It is easy to verify that the axioms of a scalar product are fulfilled. In particular, if $(f, \bar{f})_{q, \beta} = 0$ then $f = 0$ in (L^2) . In fact, $(f, \bar{f})_{q, \beta} = \sum_{n=0}^{N_f} (n!)^{1+\beta} 2^{qn} |f^{(n)}|_{ext}^2 = 0$ if and only if $|f^{(n)}|_{ext}^2 = 0$ for each $n \in \{0, \dots, N_f\}$, so $\|f\|_{(L^2)}^2 = \sum_{n=0}^{N_f} n! |f^{(n)}|_{ext}^2 = 0$.

Let $\|\cdot\|_{q, \beta}$ be the norms corresponding to scalar products $(\cdot, \cdot)_{q, \beta}$, i.e. $\|f\|_{q, \beta} = \sqrt{(f, \bar{f})_{q, \beta}}$. Denote by $(L^2)_q^\beta$ the completions of \mathcal{P}_W with respect to these norms; and set $(L^2)^\beta := \text{pr} \lim_{q \rightarrow +\infty} (L^2)_q^\beta$ (the projective limit of spaces, i.e. $(L^2)^\beta = \bigcap_q (L^2)_q^\beta$ provided by the projective limit topology, see, e.g., [2, 3] for details).

Definition 2. The spaces $(L^2)_q^\beta$ and $(L^2)^\beta$ are called parametrized Kondratiev-type spaces of regular test functions.

As is easy to see, $F \in (L^2)_q^\beta$ if and only if F can be uniquely presented as series (5) (with kernels $F^{(n)} \in \mathcal{H}_{ext}^{(n)}$) that converges in $(L^2)_q^\beta$, and

$$\|F\|_{q, \beta}^2 := \|F\|_{(L^2)_q^\beta}^2 = \sum_{n=0}^{\infty} (n!)^{1+\beta} 2^{qn} |F^{(n)}|_{ext}^2 < \infty. \quad (8)$$

Further, it is clear that for $F, G \in (L^2)_q^\beta$ the real scalar product has a form

$$(F, G)_{(L^2)_q^\beta} = \sum_{n=0}^{\infty} (n!)^{1+\beta} 2^{qn} (F^{(n)}, G^{(n)})_{ext},$$

where $F^{(n)}, G^{(n)} \in \mathcal{H}_{ext}^{(n)}$ are the kernels from decompositions (5) for F and G respectively. Finally, $F \in (L^2)^\beta$ if and only if F can be uniquely presented in form (5) and series (8) converges for each $q \in \mathbb{Z}_+$.

Proposition ([16]). *For any $\beta \in (0, 1]$ and any $q \in \mathbb{Z}$, in the same way as for $\beta = 0$ and any $q \in \mathbb{Z}_+$, the space $(L^2)_q^\beta$ is densely and continuously embedded into (L^2) .*

Taking into account this result, we can consider a chain (a parametrized regular rigging of (L^2))

$$(L^2)^{-\beta} \supset (L^2)_{-q}^{-\beta} \supset (L^2) \supset (L^2)_q^\beta \supset (L^2)^\beta, \quad (9)$$

where $(L^2)_{-q}^{-\beta}$ and $(L^2)^{-\beta} = \text{ind } \lim_{q \rightarrow +\infty} (L^2)_{-q}^{-\beta}$ (the inductive limit of spaces, i.e. $(L^2)^{-\beta} = \bigcup_q (L^2)_{-q}^{-\beta}$ provided by the inductive limit topology, see, e.g., [2, 3] for details) are the spaces dual of $(L^2)_q^\beta$ and $(L^2)^\beta$ respectively.

Definition 3. *The spaces $(L^2)_{-q}^{-\beta}$ and $(L^2)^{-\beta}$ are called parametrized Kondratiev-type spaces of regular generalized functions.*

The following statement from the definition of $(L^2)_{-q}^{-\beta}$ and the general duality theory follows.

Proposition. *1) Any regular generalized function $F \in (L^2)_{-q}^{-\beta}$ can be uniquely presented as formal series (5) (with kernels $F^{(n)} \in \mathcal{H}_{ext}^{(n)}$) that converges in $(L^2)_{-q}^{-\beta}$, and*

$$\|F\|_{-q, -\beta}^2 := \|F\|_{(L^2)_{-q}^{-\beta}}^2 = \sum_{n=0}^{\infty} (n!)^{1-\beta} 2^{-qn} |F^{(n)}|_{ext}^2 < \infty. \quad (10)$$

Vice versa, any formal series (5) such that series (10) converges, is a regular generalized function from $(L^2)_{-q}^{-\beta}$ (i.e. now series (5) converges in $(L^2)_{-q}^{-\beta}$).

2) For $F, G \in (L^2)_{-q}^{-\beta}$ the real scalar product has a form

$$(F, G)_{(L^2)_{-q}^{-\beta}} = \sum_{n=0}^{\infty} (n!)^{1-\beta} 2^{-qn} (F^{(n)}, G^{(n)})_{ext},$$

where $F^{(n)}, G^{(n)} \in \mathcal{H}_{ext}^{(n)}$ are the kernels from decompositions (5) for F and G respectively.

3) The dual pairing between $F \in (L^2)_{-q}^{-\beta}$ and $f \in (L^2)_q^\beta$ that is generated by the scalar product in (L^2) , has a form

$$\langle\langle F, f \rangle\rangle_{(L^2)} = \sum_{n=0}^{\infty} n! (F^{(n)}, f^{(n)})_{ext},$$

where $F^{(n)}, f^{(n)} \in \mathcal{H}_{ext}^{(n)}$ are the kernels from decompositions (5) for F and f respectively.

4) $F \in (L^2)^{-\beta}$ if and only if F can be uniquely presented in form (5) and norm (10) is finite for some $q \in \mathbb{Z}_+$.

Note that the term "regular generalized functions" is connected with the fact that the kernels from decompositions (5) for elements of positive and negative spaces of chain (9) belong to the same spaces $\mathcal{H}_{ext}^{(n)}$.

1.4 Stochastic integration and differentiation

In this subsection it will be convenient to denote the spaces $(L^2)_q^\beta$, $(L^2) = (L^2)_0^0$ and $(L^2)_{-q}^{-\beta}$ from chain (9) by $(L^2)_q^\beta$, $\beta \in [-1, 1]$, $q \in \mathbb{Z}$. The norms in these spaces are given, obviously, by formula (8) (cf. (8) and (10)).

Decomposition (5) for elements of $(L^2)_q^\beta$ defines an isometric isomorphism (a generalized Wiener-Itô-Sigal isomorphism)

$$\mathbf{I} : (L^2)_q^\beta \rightarrow \bigoplus_{n=0}^{\infty} (n!)^{1+\beta} 2^{qn} \mathcal{H}_{ext}^{(n)},$$

where $\bigoplus_{n=0}^{\infty} (n!)^{1+\beta} 2^{qn} \mathcal{H}_{ext}^{(n)}$ is a weighted extended symmetric Fock space (cf. [20]): for $F \in (L^2)_q^\beta$ of form (5) $\mathbf{I}F = (F^{(0)}, F^{(1)}, \dots) \in \bigoplus_{n=0}^{\infty} (n!)^{1+\beta} 2^{qn} \mathcal{H}_{ext}^{(n)}$. Let $\mathbf{1} : \mathcal{H}_C \rightarrow \mathcal{H}_C$ be the identity operator. Then the operator $\mathbf{I} \otimes \mathbf{1} : (L^2)_q^\beta \otimes \mathcal{H}_C \rightarrow \left(\bigoplus_{n=0}^{\infty} (n!)^{1+\beta} 2^{qn} \mathcal{H}_{ext}^{(n)} \right) \otimes \mathcal{H}_C \cong \bigoplus_{n=0}^{\infty} (n!)^{1+\beta} 2^{qn} (\mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}_C)$ is an isometric isomorphism between the spaces $(L^2)_q^\beta \otimes \mathcal{H}_C$ and $\bigoplus_{n=0}^{\infty} (n!)^{1+\beta} 2^{qn} (\mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}_C)$. It is obvious that for arbitrary $m \in \mathbb{Z}_+$ and $F^{(m)} \in \mathcal{H}_{ext}^{(m)} \otimes \mathcal{H}_C$ a vector $(\underbrace{0, \dots, 0}_m, F^{(m)}, 0, \dots)$ belongs to $\bigoplus_{n=0}^{\infty} (n!)^{1+\beta} 2^{qn} (\mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}_C)$. Set

$$:\langle \circ^{\otimes m}, F^{(m)} \rangle : \stackrel{def}{=} (\mathbf{I} \otimes \mathbf{1})^{-1} (\underbrace{0, \dots, 0}_m, F^{(m)}, 0, \dots) \in (L^2)_q^\beta \otimes \mathcal{H}_C.$$

It is clear that elements $:\langle \circ^{\otimes n}, F^{(n)} \rangle :$, $n \in \mathbb{Z}_+$, form orthogonal bases in the spaces $(L^2)_q^\beta \otimes \mathcal{H}_C$ in the sense that any $F \in (L^2)_q^\beta \otimes \mathcal{H}_C$ can be uniquely presented as

$$F(\cdot) = \sum_{n=0}^{\infty} :\langle \circ^{\otimes n}, F^{(n)} \rangle :, \quad F^{(n)} \in \mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}_C \quad (11)$$

(the series converges in $(L^2)_q^\beta \otimes \mathcal{H}_C$), with

$$\|F\|_{(L^2)_q^\beta \otimes \mathcal{H}_C}^2 = \sum_{n=0}^{\infty} (n!)^{1+\beta} 2^{qn} |F^{(n)}|_{\mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}_C}^2 < \infty.$$

Let us describe the construction of an extended stochastic integral that is based on decomposition (11) (a detailed presentation is given in [16, 17]). Let $F^{(n)} \in \mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}_C$, $n \in \mathbb{N}$. We select a representative (a function) $\dot{f}^{(n)} \in F^{(n)}$ such that

$$\dot{f}_u^{(n)}(u_1, \dots, u_n) = 0 \text{ if for some } k \in \{1, \dots, n\} u = u_k. \quad (12)$$

Accept on default $\Delta \in \mathcal{B}(\mathbb{R}_+)$ (we remind that \mathcal{B} denotes the Borel σ -algebra). Let $\widehat{f}_\Delta^{(n)}$ be the symmetrization of a function $\dot{f}^{(n)} 1_\Delta(\cdot)$ by $n+1$ variables. Define $\widehat{F}_\Delta^{(n)} \in \mathcal{H}_{ext}^{(n+1)}$ as the equivalence class in $\mathcal{H}_{ext}^{(n+1)}$ generated by $\widehat{f}_\Delta^{(n)}$ (i.e. $\widehat{f}_\Delta^{(n)} \in \widehat{F}_\Delta^{(n)}$). It is proved in [16, 17] that this definition is well-posed (in particular, $\widehat{F}_\Delta^{(n)}$ does not depend on a choice of a representative $\dot{f}^{(n)} \in F^{(n)}$ satisfying (12)) and $|\widehat{F}_\Delta^{(n)}|_{ext} \leq |F^{(n)} 1_\Delta(\cdot)|_{\mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}_C} \leq |F^{(n)}|_{\mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}_C}$.

Definition 4. We define the extended stochastic integral with respect to a Lévy process L

$$\int_{\Delta} \circ(u) \widehat{dL}_u : (L^2)_q^\beta \otimes \mathcal{H}_{\mathbb{C}} \rightarrow (L^2)_{q-1}^\beta \quad (13)$$

by a formula

$$\int_{\Delta} F(u) \widehat{dL}_u := \sum_{n=0}^{\infty} : \langle \circ^{\otimes n+1}, \widehat{F}_{\Delta}^{(n)} \rangle :, \quad (14)$$

where $\widehat{F}_{\Delta}^{(0)} := F^{(0)} 1_{\Delta}(\cdot) \in \mathcal{H}_{\mathbb{C}} = \mathcal{H}_{ext}^{(1)}$, and $\widehat{F}_{\Delta}^{(n)} \in \mathcal{H}_{ext}^{(n+1)}$, $n \in \mathbb{N}$, are constructed by the kernels $F^{(n)} \in \mathcal{H}_{ext}^{(n)} \otimes \mathcal{H}_{\mathbb{C}}$ from decomposition (11) for F .

One can show quite analogously to [16] that this integral is a linear *continuous* operator; and, moreover, if F is integrable by Itô then F is integrable in the extended sense and the extended stochastic integral coincides with the Itô stochastic integral.

Sometimes it can be convenient to define the extended stochastic integral by formula (14) as a linear operator

$$\int_{\Delta} \circ(u) \widehat{dL}_u : (L^2)_q^\beta \otimes \mathcal{H}_{\mathbb{C}} \rightarrow (L^2)_q^\beta. \quad (15)$$

If $\beta = -1$ then this operator is continuous (bounded) [16], for $\beta \in (-1, 1]$ operator (15) is unbounded. But if we accept the set

$$\left\{ F \in (L^2)_q^\beta \otimes \mathcal{H}_{\mathbb{C}} : \left\| \int_{\Delta} F(u) \widehat{dL}_u \right\|_{q,\beta}^2 = \sum_{n=0}^{\infty} ((n+1)!)^{1+\beta} 2^{q(n+1)} |\widehat{F}_{\Delta}^{(n)}|_{ext}^2 < \infty \right\}$$

as the domain of integral (15) then the last is a *closed* operator [16]. Also we note that the extended stochastic integral can be defined by formula (14) as a linear continuous operator acting from $(L^2)^\beta \otimes \mathcal{H}_{\mathbb{C}} := \text{pr} \lim_{q \rightarrow +\infty} (L^2)_q^\beta \otimes \mathcal{H}_{\mathbb{C}}$ to $(L^2)^\beta$, or from $(L^2)^{-\beta} \otimes \mathcal{H}_{\mathbb{C}} := \text{ind} \lim_{q \rightarrow +\infty} (L^2)_{-q}^{-\beta} \otimes \mathcal{H}_{\mathbb{C}}$ to $(L^2)^{-\beta}$, here $\beta \in [0, 1]$.

Now for plenitude of picture we recall very briefly a notion of a Hida stochastic derivative in the Lévy white noise analysis, in terms of Lytvynov's CRP (see [8, 16, 17] for a detailed presentation).

Definition 5. We define the Hida stochastic derivative $1_{\Delta}(\cdot)\partial. : (L^2)_{1-q}^{-\beta} \rightarrow (L^2)_{-q}^{-\beta} \otimes \mathcal{H}_{\mathbb{C}}$ as a linear continuous operator adjoint to extended stochastic integral (13), i.e. for all $F \in (L^2)_q^\beta \otimes \mathcal{H}_{\mathbb{C}}$ and $G \in (L^2)_{1-q}^{-\beta}$

$$\langle\langle F(\cdot), 1_{\Delta}(\cdot)\partial.G \rangle\rangle_{(L^2) \otimes \mathcal{H}_{\mathbb{C}}} = \langle\langle \int_{\Delta} F(u) \widehat{dL}_u, G \rangle\rangle_{(L^2)}.$$

If instead of integral (13) one uses integral (15), the corresponding Hida stochastic derivative will be a linear unbounded (except the case $\beta = -1$), but closed operator acting from $(L^2)_{-q}^{-\beta}$ to $(L^2)_{-q}^{-\beta} \otimes \mathcal{H}_{\mathbb{C}}$ [8]. Further, it is clear that the Hida stochastic derivative can be defined as a linear continuous operator acting from $(L^2)^\beta$ to $(L^2)^\beta \otimes \mathcal{H}_{\mathbb{C}}$ ($\beta \in [-1, 1]$) that is adjoint to the corresponding extended stochastic integral. We note also that the extended stochastic integral and the Hida stochastic derivative are mutually adjoint operators [8, 16, 17].

Let us write out an explicit formula for the Hida stochastic derivative in terms of decompositions by the Wick monomials. Let $G^{(n)} \in \mathcal{H}_{ext}^{(n)}$, $n \in \mathbb{N}$, $\dot{g}^{(n)} \in G^{(n)}$ be a representative of $G^{(n)}$.

We consider $\dot{g}^{(n)}(\cdot)$, i.e. separate a one argument of $\dot{g}^{(n)}$, and define $G^{(n)}(\cdot) \in \mathcal{H}_{ext}^{(n-1)} \otimes \mathcal{H}_{\mathbb{C}}$ as the equivalence class in $\mathcal{H}_{ext}^{(n-1)} \otimes \mathcal{H}_{\mathbb{C}}$ generated by $\dot{g}^{(n)}(\cdot)$ (i.e. $\dot{g}^{(n)}(\cdot) \in G^{(n)}(\cdot)$). It is proved in [17] that this definition is well-posed (in particular, $G^{(n)}(\cdot)$ does not depend on a choice of a representative $\dot{g}^{(n)} \in G^{(n)}$) and

$$|G^{(n)}(\cdot)|_{\mathcal{H}_{ext}^{(n-1)} \otimes \mathcal{H}_{\mathbb{C}}} \leq |G^{(n)}|_{ext}. \quad (16)$$

Note that, in spite of estimate (16), the space $\mathcal{H}_{ext}^{(n)}$, $n \in \mathbb{N} \setminus \{1\}$, is not a subspace of $\mathcal{H}_{ext}^{(n-1)} \otimes \mathcal{H}_{\mathbb{C}}$ because different elements of $\mathcal{H}_{ext}^{(n)}$ can coincide as elements of $\mathcal{H}_{ext}^{(n-1)} \otimes \mathcal{H}_{\mathbb{C}}$.

The following statement easily follows from results of [8, 16, 17].

Proposition. *For a test or square integrable or generalized function G of form (5)*

$$1_{\Delta}(\cdot) \partial . G = \sum_{n=1}^{\infty} n : \langle \circ^{\otimes n-1}, G^{(n)}(\cdot) 1_{\Delta}(\cdot) \rangle : \equiv \sum_{n=0}^{\infty} (n+1) : \langle \circ^{\otimes n}, G^{(n+1)}(\cdot) 1_{\Delta}(\cdot) \rangle :$$

At last, we recall a notion of operators of stochastic differentiation (see [6, 7] for a detailed presentation). Let $n, m \in \mathbb{Z}_+$. Consider a function $h : \mathbb{R}_+^{n+m} \rightarrow \mathbb{C}$. Denote

$$\begin{aligned} & \tilde{h}(u_1, \dots, u_n; u_{n+1}, \dots, u_{n+m}) \\ & := \begin{cases} h(u_1, \dots, u_{n+m}), & \text{if for all } i \in \{1, \dots, n\}, j \in \{n+1, \dots, n+m\} \ u_i \neq u_j \\ 0, & \text{in other cases} \end{cases} \end{aligned} \quad (17)$$

Let $F^{(n)} \in \mathcal{H}_{ext}^{(n)}$, $G^{(m)} \in \mathcal{H}_{ext}^{(m)}$. We select representatives (functions) $\dot{f}^{(n)} \in F^{(n)}$ and $\dot{g}^{(m)} \in G^{(m)}$. Set $h(u_1, \dots, u_{n+m}) := \dot{f}^{(n)}(u_1, \dots, u_n) \cdot \dot{g}^{(m)}(u_{n+1}, \dots, u_{n+m})$. Let $\widehat{f^{(n)}g^{(m)}}$ be the symmetrization of \tilde{h} (see (17)) by all variables, $F^{(n)} \diamond G^{(m)} \in \mathcal{H}_{ext}^{(n+m)}$ be the equivalence class in $\mathcal{H}_{ext}^{(n+m)}$ that is generated by $\widehat{f^{(n)}g^{(m)}}$ (i.e. $\widehat{f^{(n)}g^{(m)}} \in F^{(n)} \diamond G^{(m)}$). It is proved in [6] that this definition is well-posed (in particular, $F^{(n)} \diamond G^{(m)}$ does not depend on a choice of representatives from $F^{(n)}$ and $G^{(m)}$) and

$$|F^{(n)} \diamond G^{(m)}|_{ext} \leq |F^{(n)}|_{ext} |G^{(m)}|_{ext}. \quad (18)$$

Let $F^{(m)} \in \mathcal{H}_{ext}^{(m)}$, $f^{(n)} \in \mathcal{H}_{ext}^{(n)}$, $m > n$. We define a "product" $(f^{(n)}, F^{(m)})_{ext} \in \mathcal{H}_{ext}^{(m-n)}$ by setting for each $g^{(m-n)} \in \mathcal{H}_{ext}^{(m-n)}$

$$(g^{(m-n)}, (f^{(n)}, F^{(m)})_{ext})_{ext} = (f^{(n)} \diamond g^{(m-n)}, F^{(m)})_{ext}. \quad (19)$$

Since by the Cauchy-Bunyakovsky inequality and (18)

$$|(f^{(n)} \diamond g^{(m-n)}, F^{(m)})_{ext}| \leq |f^{(n)} \diamond g^{(m-n)}|_{ext} |F^{(m)}|_{ext} \leq |f^{(n)}|_{ext} |g^{(m-n)}|_{ext} |F^{(m)}|_{ext},$$

this definition is well-posed and

$$|(f^{(n)}, F^{(m)})_{ext}|_{ext} \leq |f^{(n)}|_{ext} |F^{(m)}|_{ext}. \quad (20)$$

Definition 6. Let $n \in \mathbb{N}$, $f^{(n)} \in \mathcal{H}_{ext}^{(n)}$. We define an operator of stochastic differentiation

$$(D^n \circ)(f^{(n)}) : (L^2)_q^\beta \rightarrow (L^2)_{q-1}^\beta \quad (21)$$

by setting for $F \in (L^2)_q^\beta$

$$\begin{aligned} (D^n F)(f^{(n)}) &:= \sum_{m=n}^{\infty} \frac{m!}{(m-n)!} : \langle \circ^{\otimes m-n}, (f^{(n)}, F^{(m)})_{ext} \rangle : \\ &\equiv \sum_{m=0}^{\infty} \frac{(m+n)!}{m!} : \langle \circ^{\otimes m}, (f^{(n)}, F^{(m+n)})_{ext} \rangle : , \end{aligned} \quad (22)$$

where $F^{(m)} \in \mathcal{H}_{ext}^{(m)}$ are the kernels from decomposition (5) for F .

Using estimate (20) one can show [6] that this definition is well-posed and operator (21) is linear and *continuous*. Moreover, in the case $\beta = 1$ formula (22) defines a linear *continuous* operator $(D^n \circ)(f^{(n)})$ on $(L^2)_q^1$, $q \in \mathbb{Z}$.

Finally, as is easily seen, $(D^n \circ)(f^{(n)})$ can be defined by formula (22) as a linear continuous operator on $(L^2)^\beta$, $\beta \in [-1, 1]$. Namely a linear continuous operator

$$(D \circ)(g) := (D^1 \circ)(g) : (L^2)^{-\beta} \rightarrow (L^2)^{-\beta}, \quad g \in \mathcal{H}_{ext}^{(1)} = \mathcal{H}_C, \quad \beta \in [0, 1], \quad (23)$$

will be a subject of study in the forthcoming section.

Properties of operators of stochastic differentiation on spaces of regular test and generalized functions of the Lévy white noise analysis are considered in detail in [6,7,9]. Here we note only that the operator $D \equiv D^1$ and the Hida stochastic derivative are connected as follows [7]. Denote $\partial. := 1_{\mathbb{R}_+}(\cdot)\partial.$. Let $F \in (L^2)_q^\beta$ and $g \in \mathcal{H}_{ext}^{(1)} = \mathcal{H}_C$. Then

$$(DF)(g) = \int_{\mathbb{R}_+} \partial_u F \cdot g(u) du \in (L^2)_{q-1}^\beta,$$

here the integral in the right hand side is a Pettis one (the weak integral). Taking into account this equality, one can write *formally*

$$\partial. \circ = (D \circ)(\delta.), \quad (24)$$

where $\delta.$ is the Dirac delta-function concentrated at $.$. In order to give a nonformal sense to equality (24), one can consider operators of stochastic differentiation on so-called spaces of nonregular generalized functions, see [18].

2 ELEMENTS OF WICK CALCULUS

2.1 Wick product and Wick versions of holomorphic functions

In this subsection we introduce and study a Wick product and Wick versions of holomorphic functions on $(L^2)^{-\beta}$, now $\beta \in [0, 1]$.

First we give necessary definitions.

Definition 7. For $F \in (L^2)^{-\beta}$ we define an S -transform $(SF)(\lambda)$, $\lambda \in \mathcal{D}_{\mathbb{C}}$, as a formal series

$$(SF)(\lambda) := \sum_{m=0}^{\infty} (F^{(m)}, \lambda^{\otimes m})_{ext} \equiv F^{(0)} + \sum_{m=1}^{\infty} (F^{(m)}, \lambda^{\otimes m})_{ext}, \quad (25)$$

where $F^{(m)} \in \mathcal{H}_{ext}^{(m)}$, $m \in \mathbb{Z}_+$, are the kernels from decomposition (5) for F (each term in series(25) is well-defined, but the series can diverge). In particular, $(SF)(0) = F^{(0)}$, $S1 \equiv 1$.

Definition 8. For $F, G \in (L^2)^{-\beta}$ and a holomorphic at $F^{(0)}$ function $h : \mathbb{C} \rightarrow \mathbb{C}$ we define a Wick product $F \diamond G$ and a Wick version $h^{\diamond}(F)$ by setting formally

$$F \diamond G := S^{-1}(SF \cdot SG), \quad h^{\diamond}(F) := S^{-1}h(SF). \quad (26)$$

Remark. It is obvious that the Wick product \diamond is commutative, associative and distributive over a field \mathbb{C} .

A function h from Definition 8 can be decomposed in a Taylor series

$$h(u) = \sum_{m=0}^{\infty} h_m (u - (SF)(0))^m. \quad (27)$$

Using this decomposition, it is easy to calculate that

$$h^{\diamond}(F) = \sum_{m=0}^{\infty} h_m (F - (SF)(0))^{\diamond m}, \quad (28)$$

where $F^{\diamond m} := \underbrace{F \diamond \dots \diamond F}_{m \text{ times}}$, $F^{\diamond 0} := 1$.

It easily follows from formula (2.23) in [19] that for $F^{(n)} \in \mathcal{H}_{ext}^{(n)}$, $G^{(m)} \in \mathcal{H}_{ext}^{(m)}$, $n, m \in \mathbb{Z}_+$, and $\lambda \in \mathcal{D}_{\mathbb{C}}$

$$(F^{(n)}, \lambda^{\otimes n})_{ext} (G^{(m)}, \lambda^{\otimes m})_{ext} = (F^{(n)} \diamond G^{(m)}, \lambda^{\otimes n+m})_{ext} \quad (29)$$

(a product \diamond is defined in Subsection 1.4).

Using this formula, by analogy with the Meixner analysis [14] one can prove the following statement.

Proposition. For $F_1, \dots, F_n \in (L^2)^{-\beta}$

$$F_1 \diamond \dots \diamond F_n = \sum_{m=0}^{\infty} : \langle \circ^{\otimes m}, \sum_{k_1, \dots, k_n \in \mathbb{Z}_+ : k_1 + \dots + k_n = m} F_1^{(k_1)} \diamond \dots \diamond F_n^{(k_n)} \rangle : \quad (30)$$

(in particular, for $F, G \in (L^2)^{-\beta}$ $F \diamond G = \sum_{m=0}^{\infty} : \langle \circ^{\otimes m}, \sum_{k=0}^m F^{(k)} \diamond G^{(m-k)} \rangle :$), where $F_j^{(k_j)} \in \mathcal{H}_{ext}^{(k_j)}$,

$j \in \{1, \dots, n\}$, $k_j \in \mathbb{Z}_+$, are the kernels from decompositions (5) for F_j ; $F^{(k)}, G^{(k)} \in \mathcal{H}_{ext}^{(k)}$, $k \in \mathbb{Z}_+$, are the kernels from the same decompositions for F and G respectively. Further, for $F \in (L^2)^{-\beta}$ and a holomorphic at $(SF)(0) = F^{(0)}$ function $h : \mathbb{C} \rightarrow \mathbb{C}$

$$h^{\diamond}(F) = h_0 + \sum_{m=1}^{\infty} : \langle \circ^{\otimes m}, \sum_{n=1}^m h_n \sum_{k_1, \dots, k_n \in \mathbb{N} : k_1 + \dots + k_n = m} F^{(k_1)} \diamond \dots \diamond F^{(k_n)} \rangle :, \quad (31)$$

where $F^{(k)} \in \mathcal{H}_{ext}^{(k)}$, $k \in \mathbb{Z}_+$, are the kernels from decomposition (5) for F , $h_n \in \mathbb{C}$, $n \in \mathbb{Z}_+$, are the coefficients from decomposition (27) for h .

It is clear that in order to give a nonformal sense to notions "the Wick product" and "the Wick version of a holomorphic function", it is necessary to study a question about convergence of series (30) and (31) in the spaces of regular generalized functions. Using estimate (18), it is possible to do it as in the Meixner analysis [14]. As a result, for the Wick product we obtain the following statement (remind that now $\beta \in [0, 1]$).

Theorem 1. *Let $F_1, \dots, F_n \in (L^2)^{-\beta}$. Then $F_1 \diamond \dots \diamond F_n \in (L^2)^{-\beta}$. Moreover, the Wick product is continuous in the topology of $(L^2)^{-\beta}$: for arbitrary $F_1, \dots, F_n \in (L^2)^{-\beta}$, $n \in \mathbb{N}$, there exist $q, q' \in \mathbb{Z}_+$ ($q > q' + (1 - \beta) \log_2 n + 1$) such that*

$$\|F_1 \diamond \dots \diamond F_n\|_{-q, -\beta} \leq \sqrt{\max_{m \in \mathbb{Z}_+} [2^{-m} (m+1)^{n-1}]} \|F_1\|_{-q', -\beta} \cdots \|F_n\|_{-q', -\beta}$$

(see (10)).

Now let us pass to consideration of the Wick versions of holomorphic functions. It follows from Theorem 1 and (28) that if $F \in (L^2)^{-\beta}$ and $h : \mathbb{C} \rightarrow \mathbb{C}$ is a polynomial then $h^\diamond(F) \in (L^2)^{-\beta}$. But for a general h the situation is more complicated: as in the Meixner analysis, the cases $\beta = 1$ and $\beta \in [0, 1)$ essentially differ. The case $\beta = 1$ is comparatively simple: by analogy with [14] we obtain

Theorem 2. *Let $F \in (L^2)^{-1}$ and a function $h : \mathbb{C} \rightarrow \mathbb{C}$ be holomorphic at $(SF)(0)$. Then $h^\diamond(F) \in (L^2)^{-1}$.*

Let now $\beta \in [0, 1)$. Since $(L^2)^{-\beta} \subset (L^2)^{-1}$, for $F \in (L^2)^{-\beta}$ and a holomorphic at $(SF)(0)$ function $h : \mathbb{C} \rightarrow \mathbb{C}$, by Theorem 2 the Wick version $h^\diamond(F)$ is a well-defined element of $(L^2)^{-1}$. But at the same time it is possible that $h^\diamond(F) \notin (L^2)^{-\beta}$, if h is not a polynomial. More exactly, we have the following result.

Theorem 3. *Let $u_0 \in \mathbb{C}$, $h : \mathbb{C} \rightarrow \mathbb{C}$ be a holomorphic at u_0 function, which is not a polynomial and is such that all coefficients h_n from the Taylor decomposition*

$$h(u) = \sum_{n=0}^{\infty} h_n (u - u_0)^n \quad (32)$$

are real and non-negative. Then for each $\beta \in [0, 1)$ one can find $F \in (L^2)^{-\beta}$ with $(SF)(0) = u_0$ such that $h^\diamond(F) \notin (L^2)^{-\beta}$.

The proof of this statement, in the same way as the proofs of Theorem 4 and Theorem 5 below, is completely analogous to the proof of the corresponding statement in the Meixner white noise analysis [14] and therefore can be omitted.

It follows from Theorem 3 that if h is not a polynomial then, generally speaking, there are no estimates for coefficients from decomposition (32), which can guarantee that for arbitrary $F \in (L^2)^{-\beta}$, $\beta \in [0, 1)$, with $(SF)(0) = u_0$, $h^\diamond(F)$ is an element of $(L^2)^{-\beta}$. Nevertheless, the following statement is valid.

Theorem 4. *Let $F = \sum_{m=0}^N : \langle \circ^{\otimes m}, F^{(m)} \rangle : \in (L^2)^{-\beta}$, $F^{(m)} \in \mathcal{H}_{ext}^{(m)}$, $N \in \mathbb{Z}_+$; and coefficients $h_n \in \mathbb{C}$, $n \in \mathbb{N}$, from the Taylor decomposition*

$$h(u) = \sum_{n=0}^{\infty} h_n (u - F^{(0)})^n$$

for a holomorphic at $F^{(0)} \in \mathbb{C}$ function $h : \mathbb{C} \rightarrow \mathbb{C}$ satisfy estimates

$$|h_n| \leq \frac{K^n}{n^{nN\frac{1-\beta}{2}}} \quad (33)$$

with some $K > 0$. Then $h^\diamond(F) \in (L^2)^{-\beta}$.

Let now $0 \leq \beta_1 < \beta_2 < 1$. We describe a sufficient condition under which $h^\diamond(F) \in (L^2)^{-\beta_2}$ for $F \in (L^2)^{-\beta_1}$ (note that if $\beta_2 = 1$ then by Theorem 2 $h^\diamond(F) \in (L^2)^{-1}$ without additional conditions).

Theorem 5. Let $0 \leq \beta_1 < \beta_2 < 1$, $F \in (L^2)^{-\beta_1}$, $h : \mathbb{C} \rightarrow \mathbb{C}$ be a holomorphic at $(SF)(0)$ function. If there exists $K > 0$ such that for arbitrary $n \in \mathbb{N}$

$$|h_n| \leq \frac{K^n}{\max_{m \in \mathbb{N}: m \geq n} \left(\frac{n^{m\frac{1-\beta_2}{2}}}{([\frac{m}{n}]!)^{n\frac{\beta_2-\beta_1}{2}}} \right)},$$

where h_n are the coefficients from decomposition (27) for h , $[\cdot]$ denotes the integer part of a number, then $h^\diamond(F) \in (L^2)^{-\beta_2}$.

By analogy with the Meixner analysis [14] one can apply the above-formulated results for study of stochastic equations with Wick-type nonlinearities.

Example. Let us consider a stochastic equation

$$X_t = X_0 + \int_0^t X_s \diamond F ds + \int_0^t X_s \diamond G \widehat{dL}_s, \quad (34)$$

where $X_0, F, G \in (L^2)^{-\beta}$, $\int_0^t X_s \diamond F ds \in (L^2)^{-1}$ is a Pettis integral (the weak integral). Applying the S -transform and solving the obtained nonstochastic equation, we obtain

$$SX_t = SX_0 \cdot \exp\{SFt + SG \int_0^t \lambda(s) ds\}.$$

Now it is sufficient to apply the inverse S -transform in order to obtain the solution of (34)

$$X_t = X_0 \diamond \exp^\diamond\{Ft + G \diamond L_t\} \in (L^2)^{-1}$$

(the fact that $X_t \in (L^2)^{-1}$ follows from Theorem 1 and Theorem 2). In order to obtain $X_t \in (L^2)^{-\beta}$, $\beta < 1$, we have to impose additional conditions. For example, let F and G be "polynomials" in the sense that their decompositions (5) contain only finite number of nonzero terms. Set $N := \max[\text{pow } F, \text{pow } G + 1]$, where $\text{pow } H$ denotes the quantity of nonzero terms in decomposition (5) for H . If there exists $K > 0$ such that for arbitrary $m \in \mathbb{N}$, where $\text{pow } H$ is the greatest number of nonzero term in decomposition,

$$\frac{m^{mN\frac{1-\beta}{2}}}{m!} \leq K^m, \quad (35)$$

then by Theorem 4 $X_t \in (L^2)^{-\beta}$ (see (33), now $h(u) = \exp(u)$, $h_m = \frac{1}{m!}$ for each $m \in \mathbb{N}$). Note that estimates (35) are fulfilled if and only if $N \leq \frac{2}{1-\beta}$, this fact is proved in [14].

2.2 Wick calculus and operators of stochastic differentiation

In this subsection we consider an interconnection between the Wick calculus and the operator of stochastic differentiation D (see (23)). In particular, we'll prove that this operator is a differentiation (satisfies the Leibniz rule) with respect to the Wick multiplication.

We define a characterization set of the space $(L^2)^{-\beta}$ in terms of the S -transform, setting $B_\beta := S(L^2)^{-\beta} \equiv \{SF : F \in (L^2)^{-\beta}\}$. It is clear that B_β is a linear space, which consists of formal series $\sum_{m=0}^{\infty} (F^{(m)}, \cdot^{\otimes m})_{ext}$ (see (25)) with the kernels $F^{(m)} \in \mathcal{H}_{ext}^{(m)}$ satisfying a condition: there exists $q \in \mathbb{Z}_+$ such that $\sum_{m=0}^{\infty} (m!)^{1-\beta} 2^{-qm} |F^{(m)}|_{ext}^2 < \infty$. It follows from Definition 8 and Theorem 1 that B_β is an algebra with respect to the pointwise multiplication.

Let $g \in \mathcal{H}_{ext}^{(1)} = \mathcal{H}_C$. We define a "directional derivative" $D_g^\diamond : B_\beta \rightarrow B_\beta$ as follows. Set for $(SF)(\cdot) = \sum_{m=0}^{\infty} (F^{(m)}, \cdot^{\otimes m})_{ext} \in B_\beta$ ($F \in (L^2)^{-\beta}$, $F^{(m)} \in \mathcal{H}_{ext}^{(m)}$ are the kernels from decomposition (5) for F)

$$(D_g^\diamond SF)(\cdot) := \sum_{m=0}^{\infty} (m+1) (F^{(m+1)}, g \diamond (\cdot^{\otimes m}))_{ext} = \sum_{m=0}^{\infty} (m+1) ((g, F^{(m+1)})_{ext}, \cdot^{\otimes m})_{ext} \in B_\beta \quad (36)$$

(see (19)). Since $S^{-1}(D_g^\diamond SF) = \sum_{m=0}^{\infty} (m+1) : \langle \circ^{\otimes m}, (g, F^{(m+1)})_{ext} \rangle : = (DF)(g) \in (L^2)^{-\beta}$ (see (22)), the operator D_g^\diamond is well-defined and the following statement is valid.

Proposition. *The operator of stochastic differentiation $(D \circ)(g)$, $g \in \mathcal{H}_{ext}^{(1)} = \mathcal{H}_C$, is the pre-image of the "directional derivative" D_g^\diamond of $S \circ$ under the S -transform, i.e. for all $F \in (L^2)^{-\beta}$*

$$(DF)(g) = S^{-1}(D_g^\diamond SF) \in (L^2)^{-\beta}. \quad (37)$$

Remark. *If we introduce on B_β a topology induced by the (inductive limit) topology of $(L^2)^{-\beta}$, then the S -transform will be a topological isomorphism between a topological algebra $(L^2)^{-\beta}$ with the Wick multiplication and a topological algebra B_β with the pointwise multiplication. Now the "directional derivative" D_g^\diamond , $g \in \mathcal{H}_{ext}^{(1)}$, is the image on B_β of the operator of stochastic differentiation $(D \circ)(g)$ on $(L^2)^{-\beta}$ (under the S -transform). Of course, $D_g^\diamond : B_\beta \rightarrow B_\beta$ is a linear continuous operator.*

The main result of this subsection is the following

Theorem 6. *The operator of stochastic differentiation D is a differentiation with respect to the Wick multiplication, i.e. for arbitrary $F, G \in (L^2)^{-\beta}$ and $g \in \mathcal{H}_{ext}^{(1)} = \mathcal{H}_C$*

$$(D(F \diamond G))(g) = (DF)(g) \diamond G + F \diamond (DG)(g) \in (L^2)^{-\beta}. \quad (38)$$

Proof. First we note that the expressions in the left hand side and in the right hand side of (38) belong to $(L^2)^{-\beta}$, this follows from the definition of operator (23) and Theorem 1. Let us prove the equality (38). By (37) and the first formula in (26)

$$\begin{aligned} (D(F \diamond G))(g) &= S^{-1}(D_g^\diamond(S(F \diamond G))) = S^{-1}(D_g^\diamond(SF \cdot SG)), \\ (DF)(g) \diamond G &= S^{-1}(S(DF)(g) \cdot SG) = S^{-1}(D_g^\diamond(SF) \cdot SG), \\ F \diamond (DG)(g) &= S^{-1}(SF \cdot S(DG)(g)) = S^{-1}(SF \cdot D_g^\diamond(SG)), \end{aligned}$$

therefore it is sufficient to prove that

$$D_g^\diamond(SF \cdot SG) = D_g^\diamond(SF) \cdot SG + SF \cdot D_g^\diamond(SG). \quad (39)$$

Let $F^{(m)}, G^{(m)} \in \mathcal{H}_{ext}^{(m)}$ be the kernels from decompositions (5) for F and G respectively. Using (25), (29) and (36), we obtain

$$\begin{aligned} (SF)(\lambda) &= \sum_{n=0}^{\infty} (F^{(n)}, \lambda^{\otimes n})_{ext}, \quad (SG)(\lambda) = \sum_{m=0}^{\infty} (G^{(m)}, \lambda^{\otimes m})_{ext}, \\ (SF)(\lambda) \cdot (SG)(\lambda) &= \sum_{n,m=0}^{\infty} (F^{(n)} \diamond G^{(m)}, \lambda^{\otimes n+m})_{ext}, \\ D_g^\diamond((SF)(\lambda) \cdot (SG)(\lambda)) &= \sum_{n,m=0}^{\infty} (n+m)(F^{(n)} \diamond G^{(m)}, g \diamond \lambda^{\otimes n+m-1})_{ext}, \\ D_g^\diamond(SF)(\lambda) &= \sum_{n=0}^{\infty} n(F^{(n)}, g \diamond \lambda^{\otimes n-1})_{ext}, \quad D_g^\diamond(SG)(\lambda) = \sum_{m=0}^{\infty} m(G^{(m)}, g \diamond \lambda^{\otimes m-1})_{ext}, \\ D_g^\diamond(SF)(\lambda) \cdot (SG)(\lambda) &= \sum_{n,m=0}^{\infty} n(F^{(n)}, g \diamond \lambda^{\otimes n-1})_{ext} (G^{(m)}, \lambda^{\otimes m})_{ext}, \\ (SF)(\lambda) \cdot D_g^\diamond(SG)(\lambda) &= \sum_{n,m=0}^{\infty} m(F^{(n)}, \lambda^{\otimes n})_{ext} (G^{(m)}, g \diamond \lambda^{\otimes m-1})_{ext}, \end{aligned}$$

here $\lambda \in \mathcal{D}_{\mathbb{C}}$. So, in order to prove (39), it is sufficient to show that for all $n, m \in \mathbb{Z}_+$

$$\begin{aligned} (n+m)(F^{(n)} \diamond G^{(m)}, g \diamond \lambda^{\otimes n+m-1})_{ext} &= n(F^{(n)}, g \diamond \lambda^{\otimes n-1})_{ext} (G^{(m)}, \lambda^{\otimes m})_{ext} \\ &+ m(F^{(n)}, \lambda^{\otimes n})_{ext} (G^{(m)}, g \diamond \lambda^{\otimes m-1})_{ext}. \end{aligned} \quad (40)$$

It is easy to see that for $n = 0$ or $m = 0$ equality (40) is valid, therefore we consider the case $n, m \in \mathbb{N}$ only.

Let us consider $(n+m)(F^{(n)} \diamond G^{(m)}, g \diamond \lambda^{\otimes n+m-1})_{ext}$. Denote by $\dot{f}^{(n)} \in F^{(n)}$ and $\dot{g}^{(m)} \in G^{(m)}$ representatives of the equivalence classes $F^{(n)}$ and $G^{(m)}$. Set $\widetilde{f^{(n)}g^{(m)}} := \dot{f}^{(n)} \cdot \dot{g}^{(m)}$ (an operation $\widetilde{}$ is defined in (17)). Let $\widehat{f^{(n)}g^{(m)}}$ be the symmetrization of $\widetilde{f^{(n)}g^{(m)}}$ with respect to all arguments. We remind that $F^{(n)} \diamond G^{(m)}$ is an equivalence class in $\mathcal{H}_{ext}^{(n+m)}$ that is generated by $\widehat{f^{(n)}g^{(m)}}$: $\widehat{f^{(n)}g^{(m)}} \in F^{(n)} \diamond G^{(m)}$. Similarly for $\lambda \in \mathcal{D}_{\mathbb{C}}$ and a representative $\dot{g} \in \underline{g}, g \in \mathcal{H}_{ext}^{(1)}$, set $\lambda^{\otimes n+m-1}g := \lambda^{\otimes n+m-1} \cdot \dot{g}$, and denote by $\widehat{\lambda^{\otimes n+m-1}g}$ the symmetrization of $\lambda^{\otimes n+m-1}g$ with respect to all arguments. Then $\widehat{\lambda^{\otimes n+m-1}g} \in g \diamond \lambda^{\otimes n+m-1}$ (an equivalence class $g \diamond \lambda^{\otimes n+m-1} \in \mathcal{H}_{ext}^{(n+m)}$ is generated by $\widehat{\lambda^{\otimes n+m-1}g}$).

Without loss of generality, one can assume that $\dot{f}^{(n)}$ and $\dot{g}^{(m)}$ are symmetric functions, and $m \geq n$. Taking this into consideration, we obtain

$$\begin{aligned} \widehat{f^{(n)}g^{(m)}}(u_1, \dots, u_n; u_{n+1}, \dots, u_{n+m}) &= \frac{n!m!}{(n+m)!} \\ \times \sum_{\substack{1 \leq p_1, \dots, p_n \leq n, n+1 \leq q_1, \dots, q_m \leq n+m \\ 0 \leq r \leq n, p_1 < \dots < p_r, p_{r+1} < \dots < p_n, q_1 < \dots < q_{n-r}, q_{n-r+1} < \dots < q_m \\ u_{p_{r+1}}, \dots, u_{p_n}, u_{q_{n-r+1}}, \dots, u_{q_m}}, &\widetilde{f^{(n)}g^{(m)}}(u_{p_1}, \dots, u_{p_r}, u_{q_1}, \dots, u_{q_{n-r}}) \end{aligned} \quad (41)$$

here for $r = n$ the argument in the right hand side of (41) is $(u_1, \dots, u_n; u_{n+1}, \dots, u_{n+m})$; for $r = 0$ this argument is $(u_{q_1}, \dots, u_{q_n}; u_1, \dots, u_n, u_{q_{n+1}}, \dots, u_{q_m})$ (see [7] for a detailed explanation).

Substituting (41) in the left hand side of (40), we obtain (see (6))

$$\begin{aligned}
& (n+m)(F^{(n)} \diamond G^{(m)}, g \diamond \lambda^{\otimes n+m-1})_{ext} = (n+m)(\widehat{f^{(n)}g^{(m)}}, \widehat{\lambda^{\otimes n+m-1}g})_{ext} \\
& = (n+m) \sum_{\substack{k, l_j, s_j \in \mathbb{N}: j=1, \dots, k, l_1 > l_2 > \dots > l_k, \\ l_1 s_1 + \dots + l_k s_k = n+m}} \frac{(n+m)!}{s_1! \dots s_k!} \left(\frac{\|p_{l_1}\|_v}{l_1!} \right)^{2s_1} \dots \left(\frac{\|p_{l_k}\|_v}{l_k!} \right)^{2s_k} \\
& \times \int_{\mathbb{R}_+^{s_1 + \dots + s_k}} \widehat{f^{(n)}g^{(m)}}(\underbrace{u_1, \dots, u_1}_{l_1}, \dots, \underbrace{u_{s_1 + \dots + s_k}, \dots, u_{s_1 + \dots + s_k}}_{l_k}) \\
& \times (\widehat{\lambda^{\otimes n+m-1}g})(\underbrace{u_1, \dots, u_1}_{l_1}, \dots, \underbrace{u_{s_1 + \dots + s_k}, \dots, u_{s_1 + \dots + s_k}}_{l_k}) du_1 \dots du_{s_1 + \dots + s_k} = \\
& = (n+m) \sum_{\substack{k, l_j, s_j \in \mathbb{N}: j=1, \dots, k, l_1 > l_2 > \dots > l_k, \\ l_1 s_1 + \dots + l_k s_k = n+m}} \frac{n!m!}{s_1! \dots s_k!} \left(\frac{\|p_{l_1}\|_v}{l_1!} \right)^{2s_1} \dots \left(\frac{\|p_{l_k}\|_v}{l_k!} \right)^{2s_k} \\
& \times \left[\int_{\mathbb{R}_+^{s_1 + \dots + s_k}} \widetilde{f^{(n)}g^{(m)}}(\underbrace{u_1, \dots, u_1}_{l_1}, \dots, \underbrace{u_{s_1 + \dots + s_k}, \dots, u_{s_1 + \dots + s_k}}_{l_k}) \right. \\
& \left. \times (\widehat{\lambda^{\otimes n+m-1}g})(\underbrace{u_1, \dots, u_1}_{l_1}, \dots, \underbrace{u_{s_1 + \dots + s_k}, \dots, u_{s_1 + \dots + s_k}}_{l_k}) du_1 \dots du_{s_1 + \dots + s_k} + \dots \right]. \tag{42}
\end{aligned}$$

We say that a collection of equal among one another arguments (e.g., (u_1, \dots, u_1)) is called a *procession*. It follows from the ordering in ascending of indexes in (41) and in (6) that processions in summands in interior sums $[\dots]$ from (42) can "tear" only so that different parts of a "torn" procession will be for different parties from ';;' processions being for one side from ';;' do not switch places; and elements in processions do not switch places. In addition, it follows from a construction of $\widetilde{f^{(n)}g^{(m)}}$ (see (17)) that summands in interior sums $[\dots]$ from (42), in which a procession is divided by ';;', are equal to zero. Another summands (if there exist for a collection k, l_j, s_j) disintegrate on groups of equal among one another integrals. These groups arise by means of transpositions of processions with equal quantity of members, which are placed before ';;' and after ';;', an equality of integrals under such transpositions from the symmetric property of a function $\widehat{\lambda^{\otimes n+m-1}g}$ follows: this symmetry gives a possibility to transpose mutually processions with equal quantity of members in the argument of $\widehat{\lambda^{\otimes n+m-1}g}$. It is clear that if there are s' processions of length l before ';;' and s'' processions of length l after ';;' then by means of mutual transpositions of these processions one can obtain $\frac{(s'+s'')!}{s'!s''!}$ equal summands.

So, nonzero terms in the last expression in (42) are "connected" with equalities

$$l_1 s_1 + \dots + l_k s_k = n + m, \tag{43}$$

that can be presented in the form

$$\begin{aligned}
& l'_1 s'_1 + \dots + l'_{k'} s'_{k'} = n, \quad l''_1 s''_1 + \dots + l''_{k''} s''_{k''} = m, \\
& k', k'', l'_1, \dots, l'_{k'}, s'_1, \dots, s'_{k'}, l''_1, \dots, l''_{k''}, s''_1, \dots, s''_{k''} \in \mathbb{N}, \\
& l'_1 > \dots > l'_{k'}, \quad l''_1 > \dots > l''_{k''}
\end{aligned} \tag{44}$$

(the first sum in (44) corresponds to first n arguments of $\widetilde{f^{(n)}g^{(m)}}$, the second sum corresponds to last m arguments) as follows. For each s_j from (43) either there exists $s'_i = s_j$ ($l'_i = l_j$) or there exists $s''_i = s_j$ ($l''_i = l_j$) or there exist s'_i and s''_w such that $s'_i + s''_w = s_j$ ($l'_i = l''_w = l_j$). Inequalities for l', l'' in (44) follow from inequalities $l_1 > \dots > l_k$ and ordering of indexes in (41) and (6) (more long processions have smaller indexes of arguments).

We will replace each group of the above-described equal among one another integrals in the right hand side of (42) by a representative multiplied by a quantity of terms in the group. Also, since the Lebesgue measure is non-atomic, we can replace here $\widetilde{f^{(n)}g^{(m)}}$ by $\dot{f}^{(n)} \cdot \dot{g}^{(m)}$ (in summands that remain elements of each procession are placed on the same side of ';'). Now, taking into account that $w^{s'+s''} = w^{s'}w^{s''}$, one can rewrite the last expression in (42) in the form

$$\begin{aligned}
& \sum_{\substack{l'_1 s'_1 + \dots + l'_{k'} s'_{k'} = n, l''_1 s''_1 + \dots + l''_{k''} s''_{k''} = m, \\ k', k'', l'_1, \dots, l'_{k'}, s'_1, \dots, s'_{k'}, l''_1, \dots, l''_{k''}, s''_1, \dots, s''_{k''} \in \mathbb{N}, \\ l'_1 > \dots > l'_{k'}, l''_1 > \dots > l''_{k''}}} \frac{n!m!(n+m)}{s'_1! \dots s'_{k'}! s''_1! \dots s''_{k''}!} \\
& \times \left(\frac{\|p_{l'_1}\|_v}{l'_1!} \right)^{2s'_1} \dots \left(\frac{\|p_{l'_{k'}}\|_v}{l'_{k'}!} \right)^{2s'_{k'}} \left(\frac{\|p_{l''_1}\|_v}{l''_1!} \right)^{2s''_1} \dots \left(\frac{\|p_{l''_{k''}}\|_v}{l''_{k''}!} \right)^{2s''_{k''}} \\
& \times \int_{\mathbb{R}_+}^{s'_1 + \dots + s'_{k'} + s''_1 + \dots + s''_{k''}} \dot{f}^{(n)}(\underbrace{u_1, \dots, u_1}_{l'_1}, \dots, \underbrace{u_{s'_1 + \dots + s'_{k'}}, \dots, u_{s'_1 + \dots + s'_{k'}}}_{l'_{k'}}) \\
& \times \dot{g}^{(m)}(\underbrace{u_{n+1}, \dots, u_{n+1}}_{l''_1}, \dots, \underbrace{u_{n+s''_1 + \dots + s''_{k''}}, \dots, u_{n+s''_1 + \dots + s''_{k''}}}_{l''_{k''}}) \\
& \times (\lambda^{\widehat{\otimes n+m-1}g})(\underbrace{u_1, \dots, u_1}_{l'_1}, \dots, \underbrace{u_{s'_1 + \dots + s'_{k'}}, \dots, u_{s'_1 + \dots + s'_{k'}}}_{l'_{k'}}) \\
& \underbrace{u_{n+1}, \dots, u_{n+1}}_{l''_1}, \dots, \underbrace{u_{n+s''_1 + \dots + s''_{k''}}, \dots, u_{n+s''_1 + \dots + s''_{k''}}}_{l''_{k''}}) \\
& \times du_1 \dots du_{s'_1 + \dots + s'_{k'}} du_{n+1} \dots du_{n+s''_1 + \dots + s''_{k''}}.
\end{aligned} \tag{45}$$

Further, the symmetrization of a function $\lambda^{\widehat{\otimes n+m-1}g}$ has a form

$$(\lambda^{\widehat{\otimes n+m-1}g})(u_1, \dots, u_{n+m}) = \frac{1}{(n+m)!} \sum_{\pi \in S_{n+m}} (\lambda^{\widehat{\otimes n+m-1}g})(u_{\pi(1)}, \dots, u_{\pi(n+m)}), \tag{46}$$

where S_{n+m} is the set of all permutations of numbers $1, \dots, n+m$. This representation can be essentially simplified if we take into account that $\lambda^{\widehat{\otimes n+m-1}g}$ is a symmetric function with respect to first $n+m-1$ arguments. Namely, consider all summands from (46) with the last argument u_{n+m} . It is clear that there are $(n+m-1)!$ such summands, because they can be obtained by arbitrary permutations of arguments u_1, \dots, u_{n+m-1} . Taking into account the above-mentioned symmetry one can conclude that all these summands are equal among one another. So, it is possible to replace them by an arbitrary representative multiplied by $(n+m-1)!$. Similarly one can group summands with the last arguments $u_{n+m-1}, u_{n+m-2}, \dots, u_1$. Substituting

multiplied by $(n + m - 1)!$ representatives of these groups of summands in (46), we obtain

$$\begin{aligned} (\lambda^{\widehat{\otimes n+m-1}}g)(u_1, \dots, u_{n+m}) &= \frac{1}{(n+m)} [(\lambda^{\widehat{\otimes n+m-1}}g)(u_1, \dots, u_{n+m}) \\ &+ (\lambda^{\widehat{\otimes n+m-1}}g)(u_{n+m}, u_1, \dots, u_{n+m-1}) + \dots + (\lambda^{\widehat{\otimes n+m-1}}g)(u_2, \dots, u_{n+m}, u_1)] \end{aligned} \quad (47)$$

(representatives of the above-described groups of summands are selected subject to consequent calculations).

Substituting (47) in (45), we obtain

$$\begin{aligned} &\sum_{\substack{l'_1 s'_1 + \dots + l'_{k'} s'_{k'} = n, l''_1 s''_1 + \dots + l''_{k''} s''_{k''} = m, \\ k', k'', l'_1, \dots, l'_{k'}, s'_1, \dots, s'_{k'}, l''_1, \dots, l''_{k''}, s''_1, \dots, s''_{k''} \in \mathbb{N}, \\ l'_1 > \dots > l'_{k'}, l''_1 > \dots > l''_{k''}}} \frac{n!m!}{s'_1! \dots s'_{k'}! s''_1! \dots s''_{k''}!} \\ &\times \left(\frac{\|p_{l'_1}\|_v}{l'_1!} \right)^{2s'_1} \dots \left(\frac{\|p_{l'_{k'}}\|_v}{l'_{k'}!} \right)^{2s'_{k'}} \left(\frac{\|p_{l''_1}\|_v}{l''_1!} \right)^{2s''_1} \dots \left(\frac{\|p_{l''_{k''}}\|_v}{l''_{k''}!} \right)^{2s''_{k''}} \\ &\times \int_{\mathbb{R}_+^{s'_1 + \dots + s'_{k'} + s''_1 + \dots + s''_{k''}}} f^{(n)}(\underbrace{u_1, \dots, u_1}_{l'_1}, \dots, \underbrace{u_{s'_1 + \dots + s'_{k'}}, \dots, u_{s'_1 + \dots + s'_{k'}}}_{l'_{k'}}) \\ &\times g^{(m)}(\underbrace{u_{n+1}, \dots, u_{n+1}}_{l''_1}, \dots, \underbrace{u_{n+s''_1 + \dots + s''_{k''}}, \dots, u_{n+s''_1 + \dots + s''_{k''}}}_{l''_{k''}}) \\ &\times [(\lambda^{\widehat{\otimes n+m-1}}g)(\underbrace{u_1, \dots, u_1}_{l'_1}, \dots, \underbrace{u_{s'_1 + \dots + s'_{k'}}, \dots, u_{s'_1 + \dots + s'_{k'}}}_{l'_{k'}}) \\ &\underbrace{u_{n+1}, \dots, u_{n+1}}_{l''_1}, \dots, \underbrace{u_{n+s''_1 + \dots + s''_{k''}}, \dots, u_{n+s''_1 + \dots + s''_{k''}}}_{l''_{k''}}) + \dots] \\ &\times du_1 \dots du_{s'_1 + \dots + s'_{k'}} du_{n+1} \dots du_{n+s''_1 + \dots + s''_{k''}}, \end{aligned} \quad (48)$$

where each next term in the sum $[\dots]$ with $n + m$ summands is obtained from the previous term by the "shift of arguments": $(\cdot_1, \dots, \cdot_{n+m-1}, \cdot_{n+m}) \rightarrow (\cdot_{n+m}, \cdot_1, \dots, \cdot_{n+m-1})$ etc. Taking into account the structure of $\lambda^{\widehat{\otimes n+m-1}}g$ (in particular, its symmetry with respect to first $n + m - 1$ arguments), the non-atomicity of the Lebesgue measure, and equalities (47) for $\lambda^{\widehat{\otimes m-1}}g$ and $\lambda^{\widehat{\otimes n-1}}g$, we can continue (48) as follows:

$$\begin{aligned} &(n+m)(F^{(n)} \diamond G^{(m)}, g \diamond \lambda^{\widehat{\otimes n+m-1}})_{ext} \\ &= \sum_{\substack{l'_1 s'_1 + \dots + l'_{k'} s'_{k'} = n, l''_1 s''_1 + \dots + l''_{k''} s''_{k''} = m, \\ k', k'', l'_1, \dots, l'_{k'}, s'_1, \dots, s'_{k'}, l''_1, \dots, l''_{k''}, s''_1, \dots, s''_{k''} \in \mathbb{N}, \\ l'_1 > \dots > l'_{k'}, l''_1 > \dots > l''_{k''}}} \frac{n!m!}{s'_1! \dots s'_{k'}! s''_1! \dots s''_{k''}!} \\ &\times \left(\frac{\|p_{l'_1}\|_v}{l'_1!} \right)^{2s'_1} \dots \left(\frac{\|p_{l'_{k'}}\|_v}{l'_{k'}!} \right)^{2s'_{k'}} \left(\frac{\|p_{l''_1}\|_v}{l''_1!} \right)^{2s''_1} \dots \left(\frac{\|p_{l''_{k''}}\|_v}{l''_{k''}!} \right)^{2s''_{k''}} \\ &\times \int_{\mathbb{R}_+^{s'_1 + \dots + s'_{k'} + s''_1 + \dots + s''_{k''}}} f^{(n)}(\underbrace{u_1, \dots, u_1}_{l'_1}, \dots, \underbrace{u_{s'_1 + \dots + s'_{k'}}, \dots, u_{s'_1 + \dots + s'_{k'}}}_{l'_{k'}}) \end{aligned}$$

$$\begin{aligned}
& \times \dot{g}^{(m)} \left(\underbrace{u_{n+1}, \dots, u_{n+1}}_{l_1''}, \dots, \underbrace{u_{n+s_1''+\dots+s_{k'}''}, \dots, u_{n+s_1''+\dots+s_{k'}''}}_{l_{k'}''} \right) \times \left[\lambda_1^{l_1''}(u_1) \cdots \lambda_{s_1'+\dots+s_{k'}'}^{l_{k'}''}(u_{s_1'+\dots+s_{k'}'}) \right. \\
& \times \left. \left((\widehat{\lambda^{\otimes m-1}g}) \left(\underbrace{u_{n+1}, \dots, u_{n+1}}_{l_1''}, \dots, \underbrace{u_{n+s_1''+\dots+s_{k'}''}, \dots, u_{n+s_1''+\dots+s_{k'}''}}_{l_{k'}''} \right) + \cdots \right) \right] \\
& \times du_1 \cdots du_{s_1'+\dots+s_{k'}'} du_{n+1} \cdots du_{n+s_1''+\dots+s_{k'}''} \\
& + \sum_{\substack{l_1' s_1'+\dots+l_{k'}' s_{k'}'=n, l_1'' s_1''+\dots+l_{k'}'' s_{k'}''=m, \\ k', k'', l_1', \dots, l_{k'}', s_1', \dots, s_{k'}', l_1'', \dots, l_{k'}'', s_1'', \dots, s_{k'}'', \in \mathbb{N}, \\ l_1' > \dots > l_{k'}', l_1'' > \dots > l_{k'}''}} \frac{n!m!}{s_1'! \cdots s_{k'}'! s_1''! \cdots s_{k'}''!} \\
& \times \left(\frac{\|p_{l_1'}\|_v}{l_1'!} \right)^{2s_1'} \cdots \left(\frac{\|p_{l_{k'}'}\|_v}{l_{k'}'!} \right)^{2s_{k'}'} \left(\frac{\|p_{l_1''}\|_v}{l_1''!} \right)^{2s_1''} \cdots \left(\frac{\|p_{l_{k'}''}\|_v}{l_{k'}''!} \right)^{2s_{k'}''} \\
& \times \int_{\mathbb{R}_+^{s_1'+\dots+s_{k'}'+s_1''+\dots+s_{k'}''}} \dot{f}^{(n)} \left(\underbrace{u_1, \dots, u_1}_{l_1'}, \dots, \underbrace{u_{s_1'+\dots+s_{k'}'}, \dots, u_{s_1'+\dots+s_{k'}'}}_{l_{k'}'} \right) \\
& \times \dot{g}^{(m)} \left(\underbrace{u_{n+1}, \dots, u_{n+1}}_{l_1''}, \dots, \underbrace{u_{n+s_1''+\dots+s_{k'}''}, \dots, u_{n+s_1''+\dots+s_{k'}''}}_{l_{k'}''} \right) \\
& \times \left[\left((\widehat{\lambda^{\otimes n-1}g}) \left(\underbrace{u_1, \dots, u_1}_{l_1'}, \dots, \underbrace{u_{s_1'+\dots+s_{k'}'}, \dots, u_{s_1'+\dots+s_{k'}'}}_{l_{k'}'} \right) + \cdots \right) \right. \\
& \times \left. \lambda_1^{l_1''}(u_{n+1}) \cdots \lambda_{s_1''+\dots+s_{k'}''}^{l_{k'}''}(u_{n+s_1''+\dots+s_{k'}''}) \right] du_1 \cdots du_{s_1'+\dots+s_{k'}'} du_{n+1} \cdots du_{n+s_1''+\dots+s_{k'}''} \\
& = m(\dot{f}^{(n)}, \lambda^{\otimes n})_{\text{ext}}(\dot{g}^{(m)}, \widehat{\lambda^{\otimes m-1}g})_{\text{ext}} + n(\dot{f}^{(n)}, \widehat{\lambda^{\otimes n-1}g})_{\text{ext}}(\dot{g}^{(m)}, \lambda^{\otimes m})_{\text{ext}} \\
& = m(F^{(n)}, \lambda^{\otimes n})_{\text{ext}}(G^{(m)}, g \diamond \lambda^{\otimes m-1})_{\text{ext}} + n(F^{(n)}, g \diamond \lambda^{\otimes n-1})_{\text{ext}}(G^{(m)}, \lambda^{\otimes m})_{\text{ext}}.
\end{aligned}$$

So, (40) is fulfilled, hence (39) is valid and therefore equality (38) is proved. \square

Corollary. Let $F \in (L^2)^{-\beta}$, $g \in \mathcal{H}_{\text{ext}}^{(1)} = \mathcal{H}_{\mathbb{C}}$, and $h : \mathbb{C} \rightarrow \mathbb{C}$ be a holomorphic at $(SF)(0)$ function. Then

$$(Dh^{\diamond}(F))(g) = h^{\diamond}(F) \diamond (DF)(g) \in (L^2)^{-1}, \quad (49)$$

where h^{\diamond} is the Wick version of the usual derivative of a function h .

Proof. First we'll prove by the mathematical induction method that for each $m \in \mathbb{Z}_+$

$$\left(D(F - (SF)(0))^{\diamond m} \right)(g) = m(F - (SF)(0))^{\diamond m-1} \diamond (DF)(g). \quad (50)$$

In fact, in the case $m = 0$ equality (50) is, obviously, true (we remind that $(F - (SF)(0))^{\diamond 0} = 1$ by definition and for $G \in \mathbb{C} \subset (L^2)^{-\beta}$ $DG = 0$). Let us suppose that (50) is valid for $m \leq k$, $k \in \mathbb{Z}_+$. In particular,

$$\left(D(F - (SF)(0))^{\diamond k} \right)(g) = k(F - (SF)(0))^{\diamond k-1} \diamond (DF)(g). \quad (51)$$

We have to show that

$$\left(D(F - (SF)(0))^{\diamond k+1} \right)(g) = (k+1)(F - (SF)(0))^{\diamond k} \diamond (DF)(g).$$

Using (38) and (51) we obtain

$$\begin{aligned}
 & \left(D (F - (SF)(0))^{\diamond k+1} \right) (g) = \left(D \left[(F - (SF)(0))^{\diamond k} \diamond (F - (SF)(0)) \right] \right) (g) \\
 & = \left(D (F - (SF)(0))^{\diamond k} \right) (g) \diamond (F - (SF)(0)) + (F - (SF)(0))^{\diamond k} \diamond \left(D (F - (SF)(0)) \right) (g) \\
 & = k (F - (SF)(0))^{\diamond k-1} \diamond (DF)(g) \diamond (F - (SF)(0)) + (F - (SF)(0))^{\diamond k} \diamond (DF)(g) \\
 & = k (F - (SF)(0))^{\diamond k} \diamond (DF)(g) + (F - (SF)(0))^{\diamond k} \diamond (DF)(g) \\
 & = (k + 1) (F - (SF)(0))^{\diamond k} \diamond (DF)(g),
 \end{aligned}$$

which is what had to be proved.

Further, consider decomposition (28) for $h^\diamond(F)$. Let $h_N^\diamond(F) := \sum_{m=0}^N h_m (F - (SF)(0))^{\diamond m}$ be the N -th partial sum of this decomposition. It follows from the linearity of D , (50), and Theorems 2 and 1 that

$$\begin{aligned}
 \left(Dh_N^\diamond(F) \right) (g) &= \sum_{m=1}^N h_m \left(D (F - (SF)(0))^{\diamond m} \right) (g) \\
 &= \sum_{m=1}^N h_m m (F - (SF)(0))^{\diamond m-1} \diamond (DF)(g) \xrightarrow{N \rightarrow \infty} h'^\diamond(F) \diamond (DF)(g)
 \end{aligned}$$

in $(L^2)^{-1}$, where h'^\diamond is the Wick version of the usual derivative of a function h . On the other hand, since $(D \circ)(g)$ is a *continuous* operator on $(L^2)^{-1}$, $\left(Dh_N^\diamond(F) \right) (g) \xrightarrow{N \rightarrow \infty} (Dh^\diamond(F)) (g)$ in $(L^2)^{-1}$. So, equality (49) is valid. \square

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Багато об'єктів Гауссівського аналізу білого шуму (простори основних і узагальнених функцій, стохастичні інтеграли та похідні, тощо) можна будувати і досліджувати у термінах так званих хаотичних розкладів, що базуються на властивості хаотичного розкладу (ВХР): грубо кажучи, кожен квадратично інтегровний відносно гауссівської міри випадкову величину можна розкласти у ряд стохастичних інтегралів Іто від невинуватих функцій. У аналізі Леві нема ВХР (крім гауссівського та пуассонівського частинних випадків). Тим не менш, існують різні узагальнення цієї властивості. Використовуючи ці узагальнення, можна будувати різні простори основних і узагальнених функцій. І у кожному випадку необхідно вводити природний добуток на просторах узагальнених функцій, та вивчати пов'язані питання. Цей добуток називається *віківським добутком*, як у гауссівському аналізі.

Конструкція віківського добутку у аналізі Леві залежить, зокрема, від обраного узагальнення ВХР. У цій статті ми маємо справу з литвинівським узагальненням ВХР та з відповідними просторами регулярних узагальнених функцій. Метою статті є увести та вивчити віківський добуток на цих просторах, та розглянути деякі пов'язані питання (віківські версії голоморфних функцій, взаємозв'язок віківського числення з операторами стохастичного диференціювання). Основні результати статті полягають у вивченні властивостей віківського добутку та віківських версій голоморфних функцій. Зокрема, ми довели, що оператор стохастичного диференціювання є диференціюванням (задовольняє правило Лейбніца) відносно віківського множення.

Ключові слова і фрази: Процес Леві, стохастичне диференціювання, віківський добуток.

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ON NONLOCAL BOUNDARY VALUE PROBLEM FOR THE EQUATION OF MOTION OF A HOMOGENEOUS ELASTIC BEAM WITH PINNED-PINNED ENDS

In the current paper, in the domain $D = \{(t, x) : t \in (0, T), x \in (0, L)\}$ we investigate the boundary value problem for the equation of motion of a homogeneous elastic beam

$$u_{tt}(t, x) + a^2 u_{xxxx}(t, x) + b u_{xx}(t, x) + c u(t, x) = 0,$$

where $a, b, c \in \mathbb{R}$, $b^2 < 4a^2c$, with nonlocal two-point conditions

$$u(0, x) - u(T, x) = \varphi(x), \quad u_t(0, x) - u_t(T, x) = \psi(x)$$

and local boundary conditions $u(t, 0) = u(t, L) = u_{xx}(t, 0) = u_{xx}(t, L) = 0$. Solvability of this problem is connected with the problem of small denominators, whose estimation from below is based on the application of the metric approach. For almost all (with respect to Lebesgue measure) parameters of the problem, we establish conditions for the solvability of the problem in the Sobolev spaces. In particular, if $\varphi \in \mathbf{H}_{q+\rho+2}$ and $\psi \in \mathbf{H}_{q+\rho}$, where $\rho > 2$, then for almost all (with respect to Lebesgue measure in \mathbb{R}) numbers a there exists a unique solution $u \in \mathbf{C}^2([0, T]; \mathbf{H}_q)$ of the problem.

Key words and phrases: nonlocal boundary value problem, homogeneous beam, small denominator, Lebesgue measure, metric approach.

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INTRODUCTION

Among nonclassical boundary value problems for partial differential equations, the problems with nonlocal conditions occupy an important place, including those that connect the values of required solutions and its derivatives in at least two boundary or interior points of the domain considered. The periodicity conditions are the simplest of these conditions.

A general definition of nonlocal conditions and their classification were introduced by Nakhushev [11].

Nonlocal problems for partial differential equations are usually ill-posed in Hadamard's sense, and their solvability (in case of a bounded domain) is connected with the problem of small denominators and is unstable with respect to small variations both in the coefficients of the problem and the parameters of the domain.

Azizbayov and Mehraliyev [1, 2] studied the nonlocal problems for the nonlinear equation of motion of a homogeneous elastic beam in a rectangle. By using the contracting mappings principle, the authors proved the existence and uniqueness of the solution in case $|\delta| \neq 1$,

where δ is the parameter of nonlocal conditions. Sabitov [13] investigated the Cauchy problem for the equation of the beam's motion with clamped ends. In particular, theorems of the existence and uniqueness of the solution in the classes of generalized and regular functions are proved.

The present paper is based on ideas close to those used in [6, 16]. The conditions of solvability of nonlocal problem in the time variable t for fourth order hyperbolic equation in a rectangle are established. The metric approach is used to find the lower bounds of small denominators appearing in the solution of the problem.

For some classes of equations and systems of equations, non-regular class of problems with nonlocal conditions with respect to a select variable t were investigated among others in [5–9, 12, 14, 15, 17–19].

1 PROBLEM STATEMENT

In the domain $D = \{(t, x) : t \in (0, T), x \in (0, L)\}$ we consider the problem for the equation of wave motion of an elastic beam:

$$P(u) \equiv u_{tt}(t, x) + a^2 u_{xxxx}(t, x) + bu_{xx}(t, x) + cu(t, x) = 0, \quad (t, x) \in D, \quad (1)$$

with local conditions of pinned-pinned ends

$$u(t, 0) = u(t, L) = u_{xx}(t, 0) = u_{xx}(t, L) = 0, \quad 0 \leq t \leq T, \quad (2)$$

and two-point nonlocal boundary conditions

$$\begin{aligned} Q_1(u) &\equiv u(0, x) - u(T, x) = \varphi(x), \quad 0 \leq x \leq L, \\ Q_2(u) &\equiv u_t(0, x) - u_t(T, x) = \psi(x), \quad 0 \leq x \leq L, \end{aligned} \quad (3)$$

where $T > 0, L > 0, a, b, c \in \mathbb{R}$, moreover $b^2 < 4a^2c$, φ and ψ are given functions.

Note that many problems about vibrations of beams and platens in structural mechanics lead to higher order differential equations than equation of rod [1, 2, 10, 13]. In particular, the equation (1) of fourth order can model the dynamic response of homogeneous beam on an elastic foundation with axial loading [3].

Below, we use the following functional spaces:

$\mathbf{H}_q = \mathbf{H}_q[0, L]$, where $q \in \mathbb{R}$, is the Sobolev space of all trigonometric series $\varphi(x) = \sum_{k \in \mathbb{N}} \varphi_k \sin \frac{k\pi x}{L}$ with the norm $\|\varphi\|_{\mathbf{H}_q} = \left(\sum_{k \in \mathbb{N}} k^{2q} |\varphi_k|^2 \right)^{1/2}$;
 $\mathbf{C}^n([0, T]; \mathbf{H}_q)$, $n \in \mathbb{Z}_+$, is the space of all series

$$u(t, x) = \sum_{k \in \mathbb{N}} u_k(t) \sin \frac{k\pi x}{L},$$

where $u_k \in \mathbf{C}^n[0, T]$, $k \in \mathbb{N}$, with the norm

$$\|u\|_{\mathbf{C}^n([0, T]; \mathbf{H}_q)}^2 = \sum_{j=0}^n \max_{t \in [0, T]} \left\| \sum_{k \in \mathbb{N}} u_k^{(j)}(t) \sin \frac{k\pi x}{L} \right\|_{\mathbf{H}_q}^2.$$

A function $u = u(t, x)$ from $\mathbf{C}^2([0, T]; \mathbf{H}_q)$ is called a solution of the problem (1)–(3), if it satisfies conditions

$$\|P(u)\|_{\mathbf{C}^2([0, T]; \mathbf{H}_q)} = 0, \quad \|Q_1(u) - \varphi\|_{\mathbf{H}_q} = 0, \quad \|Q_2(u) - \psi\|_{\mathbf{H}_{q-2}} = 0.$$

2 THE CONDITIONS OF UNIQUENESS OF THE SOLUTION

We seek a solution of the problem (1)–(3) in the form of a series

$$u(t, x) = \sum_{k \in \mathbb{N}} u_k(t) \sin \lambda_k x, \quad (4)$$

where $\lambda_k = \frac{\pi k}{L}$. Each function $u_k(t)$, $k \in \mathbb{N}$, in (4) is a solution of the problem

$$u_k''(t) + (a^2 \lambda_k^4 - b \lambda_k^2 + c) u_k(t) = 0, \quad (5)$$

$$u_k(0) - u_k(T) = \varphi_k, \quad u_k'(0) - u_k'(T) = \psi_k, \quad (6)$$

where φ_k and ψ_k are the Fourier coefficients of the functions φ and ψ (in the system $\{\sin \lambda_k x\}$, $k \in \mathbb{N}$).

The characteristic determinant Δ_k of the problem (5), (6) is defined by

$$\Delta_k = 2\beta_k(1 - \cos \beta_k T),$$

where $\beta_k = \sqrt{a^2 \lambda_k^4 - b \lambda_k^2 + c}$, $k \in \mathbb{N}$.

Denote $\mathbb{N}_0 = \{k \in \mathbb{N} : \Delta_k = 0\}$. If $\varphi_k = 0$ and $\psi_k = 0$ for $k \in \mathbb{N}_0$, then the solution of the problem (5), (6) is not unique and it is defined by

$$u_k(t) = C_{1k} \cos \beta_k t + C_{2k} \sin \beta_k t, \quad (7)$$

where C_{1k}, C_{2k} are arbitrary constants, $k \in \mathbb{N}_0$.

If $k \in \mathbb{N} \setminus \mathbb{N}_0$, then C_{1k}, C_{2k} are solutions of the homogeneous systems of linear equations

$$\begin{cases} C_{1k}(1 - \cos \beta_k T) - C_{2k} \sin \beta_k T = 0, \\ C_{1k} \beta_k \sin \beta_k T + C_{2k}(1 - \cos \beta_k T) = 0, \end{cases}$$

and functions $u_k(t)$ have the form

$$u_k(t) = \frac{\beta_k (\cos \beta_k t - \cos \beta_k(t - T)) \varphi_k + (\sin \beta_k t + \sin \beta_k(t + T)) \psi_k}{2\beta_k(1 - \cos \beta_k T)}. \quad (8)$$

If $\beta_k T \neq 2\pi m$ for all $(k, m) \in \mathbb{N}^2$, then set \mathbb{N}_0 is empty and functions $u_k(t)$ are defined by the formula (8).

Theorem 1. For uniqueness of the solution $u \in C^2([0, T]; \mathbf{H}_q)$ of the problem (1)–(3) it is necessary and sufficient that the following condition should be satisfied:

$$(\text{for all } (k, m) \in \mathbb{N}^2) \quad \beta_k T \neq 2\pi m. \quad (9)$$

The proof follows from uniqueness of Fourier expansion of the functions with respect to the system $\{\sin \lambda_k x\}$, $k \in \mathbb{N}$.

Thus, if $\varphi_k = \psi_k = 0$ for all $k \in \mathbb{N}_0$ then from (4), (7), (8) follows that the problem (1)–(3) has the formal solution defined by

$$\begin{aligned} u(t, x) &= \sum_{k \in \mathbb{N}_0} (C_{1k} \cos \beta_k t + C_{2k} \sin \beta_k t) \sin \lambda_k x \\ &+ \sum_{k \in \mathbb{N} \setminus \mathbb{N}_0} \frac{\beta_k (\cos \beta_k t - \cos \beta_k(t - T)) \varphi_k + (\sin \beta_k t + \sin \beta_k(t + T)) \psi_k}{2\beta_k(1 - \cos \beta_k T)} \sin \lambda_k x. \end{aligned} \quad (10)$$

If conditions (9) are satisfied then the set \mathbb{N}_0 is empty and formal solution of the problem (1)–(3) is unique and is given as follows

$$u(t, x) = \sum_{k \in \mathbb{N}} \frac{\beta_k (\cos \beta_k t - \cos \beta_k(t - T)) \varphi_k + (\sin \beta_k t + \sin \beta_k(t + T)) \psi_k}{2\beta_k(1 - \cos \beta_k T)} \sin \lambda_k x. \quad (11)$$

3 THE CONDITIONS OF EXISTENCE OF THE SOLUTION

In what follows, we assume that conditions (9) are satisfied. Then there exists a unique solution of the problem (1)–(3), which admits a representation (11).

The existence of the solution $u \in \mathbf{C}^2([0, T]; \mathbf{H}_q)$ of the problem (1)–(3) is associated with the problem of small denominators, because the terms of the sequence $\{1 - \cos \beta_k T\}_{k \in \mathbb{N}}$ in the denominator of the formula (11), being different from zero, can rapidly approach zero if $k \rightarrow +\infty$. This leads to a divergence of series (11) in the space $\mathbf{C}^2([0, T]; \mathbf{H}_q)$ and therefore we get unsolvable problem in preset scale.

If we can estimate below small denominators $1 - \cos \beta_k T$ with certain or exponential behavior with respect to k , then the relevant restrictions on the functions φ and ψ we can obtain the correct solvability of the problem.

Lemma 1. *For all $k \in \mathbb{N}$, the following estimates hold*

$$C_1 k^2 \leq \beta_k \leq C_2 k^2, \quad (12)$$

where C_1, C_2 are constants independent of k .

Proof. Let r_1, r_2 are complex roots of the equation $a^2 r^2 - br + c = 0$. Then we get

$$\begin{aligned} \beta_k^2 &= a^2 |\lambda_k^2 - r_1| |\lambda_k^2 - r_2| \geq a^2 \left| \lambda_k^2 - |r_1| \right| \left| \lambda_k^2 - |r_2| \right| \\ &\geq a^2 \left| \frac{1}{2} \lambda_k^2 + \frac{1}{2} \lambda_k^2 - |r_1| \right| \left| \frac{1}{2} \lambda_k^2 + \frac{1}{2} \lambda_k^2 - |r_2| \right| \geq \frac{a^2}{4} \lambda_k^4 = C_3 k^4 \end{aligned}$$

for all $k \geq K_1$, where $C_3 = (a/2)^2 (\pi/L)^4$ and $K_1 = \pi^{-1} \sqrt{2L \max\{|r_1|, |r_2|\}}$.

If $k < K_1$, then $\beta_k^2 \geq C_4 k^4$, where $C_4 = \min_{k < K_1} \{\beta_k^2 k^{-4}\}$. Thus, $\beta_k \geq C_1 k^2$, for all $k \in \mathbb{N}$, where $C_1 = \min\{\sqrt{C_3}, \sqrt{C_4}\}$.

Similarly, we obtain the estimate from above

$$\beta_k^2 \leq a^2 \lambda_k^4 + |b| \lambda_k^4 + c \lambda_k^4 \leq C_5 k^4,$$

where $C_5 = (a^2 + |b| + c) \left(\frac{\pi}{L}\right)^4$. Therefore, $\beta_k \leq C_2 k^2$, where $C_2 = \sqrt{C_5}$. \square

Using Lemma 1, we prove existence of the solution of the problem (1)–(3).

Theorem 2. *Let conditions (9) are satisfied and there exist numbers $C_6 > 0$ and $\gamma \in \mathbb{R}$ such that inequality*

$$|1 - \cos \beta_k T| \geq C_6 k^{-\gamma} \quad (13)$$

is fulfilled for all (except finitely many numbers) $k \in \mathbb{N}$. If $\varphi \in \mathbf{H}_{q+\gamma+4}$ and $\psi \in \mathbf{H}_{q+\gamma+2}$, then exists a unique solution $u \in \mathbf{C}^2([0, T]; \mathbf{H}_q)$ of the problem (1)–(3), which continuously depends on functions φ and ψ , i.e.,

$$\|u\|_{\mathbf{C}^2([0, T]; \mathbf{H}_q)}^2 \leq C_7 \left(\|\varphi\|_{\mathbf{H}_{q+\gamma+4}}^2 + \|\psi\|_{\mathbf{H}_{q+\gamma+2}}^2 \right),$$

where C_7 is a constant independent of k .

Proof. From (8), (12), (13), we obtain the estimates

$$\left| u_k^{(j)}(t) \right| \leq \frac{C_2 C_2^j k^{2j+2} |\varphi_k| + C_2^j k^{2j} |\psi_k|}{C_1 C_6 k^{2-\gamma}} \leq C_8 k^{2j+\gamma-2} \left(k^2 |\varphi_k| + |\psi_k| \right),$$

for $j \in \{0, 1, 2\}$, where $C_8 = \frac{\max\{1, C_2^3\}}{C_1 C_6}$.

In view of the above estimate, we obtain the estimate for norm of the solution of the problem (1)–(3):

$$\begin{aligned} \|u\|_{\mathbf{C}^2([0,T];\mathbf{H}_q)}^2 &= \sum_{j=0}^2 \max_{t \in [0,T]} \left\| \sum_{k \in \mathbb{N}} u_k^{(j)}(t) \sin \lambda_k x \right\|_{\mathbf{H}_q}^2 \\ &\leq \sum_{j=0}^2 \sum_{k \in \mathbb{N}} k^{2q} C_8^2 k^{2(2j+\gamma-2)} \left(k^2 |\varphi_k| + |\psi_k| \right)^2 \\ &= 2C_8^2 \sum_{j=0}^2 \sum_{k \in \mathbb{N}} k^{2(2j+q+\gamma-2)} \left(k^4 |\varphi_k|^2 + |\psi_k|^2 \right) \\ &\leq 6C_8^2 \sum_{k \in \mathbb{N}} k^{2(q+\gamma+2)} \left(k^4 |\varphi_k|^2 + |\psi_k|^2 \right) \\ &= 6C_8^2 \sum_{k \in \mathbb{N}} \left(k^{2(q+\gamma+4)} |\varphi_k|^2 + k^{2(q+\gamma+2)} |\psi_k|^2 \right) = C_7 \left(\|\varphi\|_{\mathbf{H}_{q+\gamma+4}}^2 + \|\psi\|_{\mathbf{H}_{q+\gamma+2}}^2 \right). \end{aligned}$$

This completes the proof. \square

4 THE METRIC ESTIMATES OF THE SMALL DENOMINATORS

In Theorem 2, the condition (13) is imposed axiomatically. Now we study the conditions of validity of the inequalities (13).

To this end, we use the following assertion. Let $\text{meas } A$ is the Lebesgue measure of a measurable set A .

Lemma 2 (Borel–Cantelli [4]). *Let $\{A_k\}_{k \in \mathbb{N}}$ be a countable collection of measurable sets such that $\sum_{k \in \mathbb{N}} \text{meas } A_k < \infty$. Then the set of points that belongs to an infinite number of the sets A_k has zero Lebesgue measure.*

Lemma 3 ([16]). *Let $f \in C^1[t_1, t_2]$. If $|f'(t)| \geq \delta > 0$ for all $t \in [t_1, t_2]$, then*

$$\text{meas}\{t \in [t_1, t_2] : |f(t)| < \varepsilon\} \leq \frac{2\varepsilon}{\delta},$$

where $\varepsilon > 0$.

4.1 The estimates in terms of coefficients a, b, c .

Let $a \in [a_1, a_2]$, $0 < a_1 < a_2$, and b, c are arbitrary fixed numbers in (1).

Theorem 3. *For almost all (with respect to Lebesgue measure in \mathbb{R}) numbers $a \in [a_1, a_2]$ inequality*

$$|1 - \cos \beta_k T| \geq 2k^{-\gamma} \tag{14}$$

is fulfilled for all (except finitely many numbers) $k \in \mathbb{N}$ and $\gamma > 0$.

Proof. Since $1 - \cos \beta_k T = 2 \sin^2 \frac{\beta_k T}{2}$, it follows that establishment of estimate (14) is reduced to establishment of estimate underneath for sequence

$$f_k(a) = \sin \frac{\beta_k T}{2}, \quad k \in \mathbb{N},$$

where $\beta_k = \beta_k(a) = \sqrt{a^2 \lambda_k^4 - b \lambda_k^2 + c}$, $a \in [a_1, a_2]$.

For every fixed $k \in \mathbb{N}$ we introduce the set $A_k = \{a \in [a_1, a_2] : |f_k(a)| < k^{-\theta}, \theta > 0\}$. Let A denote the set of numbers which are infinite number of sets A_k , $k \in \mathbb{N}$. We find the estimate for measure of set A_k : for $k > 1$,

$$|f'_k(a)| = \frac{a T \lambda_k^4}{2 \beta_k} \sqrt{1 - \sin^2 \frac{\beta_k T}{2}} \geq \frac{a_1 T \lambda_k^4}{2 \beta_k} \sqrt{1 - k^{-2\theta}} \geq \frac{a_1 T \left(\frac{k\pi}{L}\right)^4}{2 C_2 k^2} \sqrt{1 - 2^{-2\theta}} = C_9 k^2,$$

where $C_9 = \frac{a_1 T}{2 C_2} \left(\frac{\pi}{L}\right)^4 \sqrt{1 - 2^{-2\theta}} > 0$. Hence, for every $k > 1$, $|f'_k(a)| \geq C_9 k^2$. Then for every $k > 1$, by Lemma 3, we obtain the estimate for measure of set A_k as follows

$$\text{meas } A_k \leq \frac{2k^{-\theta}}{C_9 k^2} = \frac{2}{C_9 k^{2+\theta}}.$$

For fixed $\theta > 0$, series $\sum_{k>1} \text{meas } A_k$ is majorized by the convergent series $\frac{2}{C_9} \sum_{k>1} \frac{1}{k^{2+\theta}}$. Then from Lemma 2 we conclude that Lebesgue measure of the set of points $a \in [a_1, a_2]$, which contained the infinite numbers of sets A_k , is equal to zero.

Thus, $\text{meas } A = 0$. So, for $a \in [a_1, a_2] \setminus A$ there exists number $k = K(a)$, that the estimate $|f_k(a)| \geq k^{-\theta}$ is valid for all $k \geq K(a)$ and $\theta > 0$. Then $|1 - \cos \beta_k T| \geq 2k^{-\gamma}$, where $\gamma = 2\theta$. \square

Theorem 4. For almost all (with respect to Lebesgue measure in \mathbb{R}) numbers $a \in [a_1, a_2]$ conditions (9) are fulfilled.

Proof. Let us consider the set $B_k = \{a \in [a_1, a_2] : \cos \beta_k(a) T = 1\}$, for fixed k . Since $\beta_k(a) = \sqrt{a^2 \lambda_k^4 - b \lambda_k^2 + c}$, we have

$$a^2 \lambda_k^4 - b \lambda_k^2 + c = \left(\frac{2\pi m}{T}\right)^2, \quad m \in \mathbb{N}. \quad (15)$$

Since equation (1) is hyperbolic, we conclude that equation (15) can not have more than two real roots relatively variable a for fixed k .

Let B_k^m be the set of roots of the equation (15). Obviously, that

$$B_k = \bigcup_{m \in \mathbb{N}} B_k^m, \quad \text{meas } B_k \leq \sum_{m \in \mathbb{N}} \text{meas } B_k^m.$$

Since $\text{meas } B_k^m = 0$ we conclude that $B_k = 0$. We introduce the set

$$B = \left\{ a \in [a_1, a_2] : \prod_{k \in \mathbb{N}} (1 - \cos \beta_k(a) T) = 0 \right\}.$$

As well as $B = \bigcup_{k \in \mathbb{N}} B_k$, it follows that $\text{meas } B = 0$.

Therefore, $\beta_k T \neq 2\pi m$, $m \in \mathbb{N}$, for almost all $a \in [a_1, a_2]$. \square

Corollary 1. For almost all (with respect to Lebesgue measure in \mathbb{R}) numbers $a \in [a_1, a_2]$, inequality

$$|1 - \cos \beta_k T| \geq C_{10} k^{-\gamma}, \quad C_{10} > 0, \quad (16)$$

holds for $\gamma > 0$ and $k \in \mathbb{N}$.

Proof. If $a \in [a_1, a_2] \setminus (A \cup B)$ then from Theorems 3 and 4 we obtain

$$|1 - \cos \beta_k T| \geq 2k^{-\gamma}, \quad k \geq K(a), \quad (17)$$

$$|1 - \cos \beta_k T| \geq k^{-\gamma} \min_{k < K(a)} \left| \frac{1 - \cos \beta_k T}{k^{-\gamma}} \right| \quad (18)$$

$$\geq C_{11} k^{-\gamma} > 0, \quad k < K(a). \quad (19)$$

Thus, $|1 - \cos \beta_k T| \geq C_{10} k^{-\gamma}$, where $C_{10} = \min\{2, C_{11}\}$, $\gamma > 0$. Note that constant C_{10} depends on a . \square

Theorem 5. For almost all (with respect to Lebesgue measure in \mathbb{R}) numbers $b \in [b_1, b_2]$ inequality

$$|1 - \cos \beta_k T| \geq C_{12} k^{-\gamma}$$

is fulfilled for $\gamma > 2$ and all $k \in \mathbb{N}$, where C_{12} is a constant independent of k .

Theorem 6. For almost all (with respect to Lebesgue measure in \mathbb{R}) numbers $c \in [c_1, c_2]$ inequality

$$|1 - \cos \beta_k T| \geq C_{13} k^{-\gamma}$$

is fulfilled for $\gamma > 6$ and all $k \in \mathbb{N}$, where C_{13} is a constant independent of k .

4.2 The estimate in terms of parameter T .

Lemma 4 ([12]). Let $\Phi(k)$ is bounded sequence of real numbers. Then for almost all (with respect to Lebesgue measure in \mathbb{R}) numbers $d > 0$ inequality

$$\left| \Phi(k) - \frac{md}{|k|^\sigma} \right| < \frac{1}{|k|^{1+\sigma+\varepsilon}},$$

where $0 < \varepsilon < 1$, $\sigma > 0$, has no more than a finitely many integer solutions in $k \neq 0$ and $m \neq 0$.

Theorem 7. For almost all (with respect to Lebesgue measure in \mathbb{R}) numbers $T > 0$ inequality

$$|1 - \cos \beta_k T| \geq 2T^2 k^{-\gamma}$$

is fulfilled for $\gamma > 2$ and all (except finitely many) numbers $k \in \mathbb{N}$.

Proof. Since $1 - \cos \beta_k T = 2 \sin^2 \beta_k \frac{T}{2}$ and $|\sin y| \geq \frac{2}{\pi} |y|$ for $|y| \leq \frac{\pi}{2}$, we can apply Lemma 4. Then

$$|1 - \cos \beta_k T| \geq \frac{8}{\pi^2} \left| \beta_k \frac{T}{2} - \pi m \right|^2 = 2T^2 k^4 \left| \frac{\beta_k}{k^2 \pi} - \frac{2m}{Tk^2} \right|^2 \geq \frac{2T^2 k^4}{|k|^{2(3+\varepsilon)}} = \frac{2T^2}{k^{2+2\varepsilon}},$$

where $m \in \mathbb{N}$ such that $|\beta_k \frac{T}{2} - \pi m| \leq \frac{\pi}{2}$, $0 < \varepsilon < 1$, $\gamma = 2 + 2\varepsilon > 2$.

Thus, $|1 - \cos \beta_k T| \geq 2T^2 k^{-\gamma}$, for $\gamma = 2(\theta + \varepsilon) > 2$ and all (except finitely many) numbers $k \in \mathbb{N}$. \square

Theorem 8. For almost all (with respect to Lebesgue measure in \mathbb{R}) numbers $T > 0$ condition (9) is fulfilled.

Corollary 2. For almost all (with respect to Lebesgue measure in \mathbb{R}) numbers $T > 0$ inequality

$$|1 - \cos \beta_k T| \geq C_{14} k^{-\gamma}$$

is fulfilled for $\gamma > 2$ and all numbers $k \in \mathbb{N}$, where C_{14} is a constant independent of k .

Corollary 3. If $\varphi \in \mathbf{H}_{q+\rho+2}$ and $\psi \in \mathbf{H}_{q+\rho}$, where $\rho > 2$, then for almost all (with respect to Lebesgue measure in \mathbb{R}) numbers $a \in [a_1, a_2]$ there exists a unique solution $u \in \mathbf{C}^2([0, T]; \mathbf{H}_q)$ of the problem (1)–(3).

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В області $D = \{(t, x) : t \in (0, T), x \in (0, L)\}$ досліджено крайову задачу для рівняння руху однорідної еластичної балки

$$u_{tt}(t, x) + a^2 u_{xxxx}(t, x) + bu_{xx}(t, x) + cu(t, x) = 0,$$

де $a, b, c \in \mathbb{R}$, $b^2 < 4a^2c$, з нелокальними двоточковими умовами

$$u(0, x) - u(T, x) = \varphi(x), \quad u_t(0, x) - u_t(T, x) = \psi(x)$$

і локальними крайовими умовами $u(t, 0) = u(t, L) = u_{xx}(t, 0) = u_{xx}(t, L) = 0$. Розв'язність цієї задачі пов'язана з проблемою малих знаменників, для оцінки знизу яких застосовується метричний підхід. Для майже всіх (стосовно міри Лебега) параметрів задачі встановлено умови розв'язності задачі в просторах Соболева. Зокрема, якщо $\varphi \in \mathbf{H}_{q+\rho+2}$ і $\psi \in \mathbf{H}_{q+\rho}$, де $\rho > 2$, то для майже всіх (стосовно міри Лебега в \mathbb{R}) чисел a існує єдиний розв'язок $u \in \mathbf{C}^2([0, T]; \mathbf{H}_q)$ задачі.

Ключові слова і фрази: нелокальна крайова задача, однорідна балка, малий знаменник, міра Лебега, метричний підхід.



KACHANOVSKY N.A.

ON WICK CALCULUS ON SPACES OF NONREGULAR GENERALIZED FUNCTIONS OF LÉVY WHITE NOISE ANALYSIS

Development of a theory of test and generalized functions depending on infinitely many variables is an important and actual problem, which is stipulated by requirements of physics and mathematics. One of successful approaches to building of such a theory consists in introduction of spaces of the above-mentioned functions in such a way that the dual pairing between test and generalized functions is generated by integration with respect to some probability measure. First it was the Gaussian measure, then it were realized numerous generalizations. In particular, important results can be obtained if one uses the Lévy white noise measure, the corresponding theory is called the *Lévy white noise analysis*.

In the Gaussian case one can construct spaces of test and generalized functions and introduce some important operators (e.g., stochastic integrals and derivatives) on these spaces by means of a so-called *chaotic representation property* (CRP): roughly speaking, any square integrable random variable can be decomposed in a series of repeated Itô's stochastic integrals from nonrandom functions. In the Lévy analysis there is no the CRP, but there are different generalizations of this property.

In this paper we deal with one of the most useful and challenging generalizations of the CRP in the Lévy analysis, which is proposed by E. W. Lytvynov, and with corresponding spaces of nonregular generalized functions. The goal of the paper is to introduce a natural product (a Wick product) on these spaces, and to study some related topics. Main results are theorems about properties of the Wick product and of Wick versions of holomorphic functions. In particular, we prove that an operator of stochastic differentiation satisfies the Leibniz rule with respect to the Wick multiplication. In addition we show that the Wick products and the Wick versions of holomorphic functions, defined on the spaces of regular and nonregular generalized functions, constructed by means of Lytvynov's generalization of the CRP, coincide on intersections of these spaces.

Our research is a contribution in a further development of the Lévy white noise analysis.

Key words and phrases: Lévy process, Wick product, stochastic differentiation.

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INTRODUCTION

Development of a theory of test and generalized functions depending on infinitely many variables (i.e., with arguments belonging to infinite-dimensional spaces) is an important and actual problem, which is stipulated by requirements of physics and mathematics (in particular, of the quantum field theory, of the mathematical physics, of the theory of random processes). A successful (but, of course, not the only) approach to building of such a theory consists in introduction of spaces of the above-mentioned functions in such a way that the dual pairing

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between test and generalized functions is generated by integration with respect to some probability measure. First it was the Gaussian measure, the corresponding theory is called the *Gaussian white noise analysis* (e.g., [7, 19, 33, 35, 37]), then it were realized numerous generalizations. In particular, important results can be obtained if one uses the Lévy white noise measure (e.g., [10, 11, 38]), the corresponding theory is called the *Lévy white noise analysis*.

In the Gaussian white noise analysis one can construct spaces of test and generalized functions and introduce some important operators (e.g., stochastic integrals and derivatives) on these spaces by means of a so-called *chaotic representation property* (CRP). This property consists, roughly speaking, in the following: any square integrable random variable can be decomposed in a series of repeated Itô's stochastic integrals from nonrandom functions (see, e.g., [39] for a detailed presentation). In the Lévy white noise analysis there is no the CRP (more exactly, the only Lévy processes with the CRP are Wiener and Poisson processes) [44]; but there are different generalizations of this property: Itô's generalization [21], Nualart-Schoutens' generalization [40, 41], Lytvynov's generalization [38], Oksendal's generalization [10, 11], etc. The interconnections between these generalizations are described in, e.g., [4, 10, 11, 29, 38, 43, 45]. Now, depending on problems under consideration, one can select a most suitable generalization of the CRP and construct corresponding spaces of test and generalized functions.

In this paper we deal with one of the most useful and challenging generalizations of the CRP in the Lévy analysis, which is proposed by E. W. Lytvynov [38] (see also [9]). The idea of this generalization is to decompose square integrable with respect to the Lévy white noise measure random variables in series of special orthogonal functions (see Subsection 1.2), by analogy with decompositions of square integrable random variables by Hermite polynomials in the Gaussian analysis (remind that the last decompositions are equivalent to decompositions by repeated stochastic integrals). In a sense, the most natural spaces that can be constructed using Lytvynov's generalization of the CRP, are spaces of *regular* test and generalized functions [25]. In a moment these spaces are well studied. In particular, the extended stochastic integral and the Hida stochastic derivative on them are introduced and studied in [14, 25], operators of stochastic differentiation — in [12, 13, 16], some elements of a Wick calculus — in [15]. But, as in the Gaussian analysis, in connection with some problems of the mathematical physics and of the stochastic analysis (in particular, of the theory of stochastic equations with Wick-type nonlinearities), it is necessary to introduce into consideration so-called spaces of *nonregular* test and generalized functions in terms of Lytvynov's generalization of the CRP [25], and to study operators and operations on these spaces. Note that, as distinct from the Gaussian analysis, now the spaces of regular generalized functions are not embedded into the spaces of nonregular generalized functions, and, accordingly, the spaces of nonregular test functions are not embedded into the spaces of regular test functions. Moreover, one can widen the extended stochastic integral from the space of square integrable random variables to the spaces of nonregular generalized functions, and, accordingly, to restrict the Hida stochastic derivative and the operators of stochastic differentiation to the spaces of nonregular test functions; but the extended stochastic integral cannot be naturally restricted to the spaces of nonregular test functions, and, accordingly, it is impossible to widen in a natural way the Hida stochastic derivative and the operators of stochastic differentiation to the spaces of nonregular generalized functions. Therefore it is necessary to introduce and to study natural analogs of the above-mentioned operators on the corresponding spaces. The stochastic integrals, derivatives, operators of stochastic differentiation, and their analogs on the spaces of nonregular test

and generalized functions are studied in detail in [25, 30, 31]. The goal of the present paper is to make the next natural step — to introduce a natural product (a Wick product) on the spaces of nonregular generalized functions, by analogy with the Gaussian analysis [34] and with the Lévy analysis on the spaces of regular generalized functions [15], and to study some related topics (Wick versions of holomorphic functions, an interconnection between the Wick calculus and the operators of stochastic differentiation). Main results of the paper are theorems about properties of the Wick product and of the Wick versions of holomorphic functions. In particular, we prove that, as in the regular case, the operator of stochastic differentiation is a differentiation (satisfies the Leibniz rule) with respect to the Wick multiplication. In addition we show that the Wick products and the Wick versions of holomorphic functions, defined on the spaces of regular and nonregular generalized functions, constructed by means of Lytvynov's generalization of the CRP, coincide on intersections of these spaces.

Note that some results of the paper can be transferred to weighted symmetric Fock spaces, by analogy with [32]. This gives an opportunity to extend an area of possible applications of these results. In particular, one can transfer them to any spaces isomorphic to the above-mentioned Fock spaces.

The paper is organized in the following manner. In the first section we introduce a Lévy process L and construct a probability triplet connected with L , convenient for our considerations; then we describe Lytvynov's generalization of the CRP; and construct a nonregular rigging of the space of square integrable random variables (the positive and negative spaces of this rigging are the spaces of nonregular test and generalized functions respectively). The second section is devoted to the Wick calculus: in the first subsection we introduce and study the Wick product and the Wick versions of holomorphic functions on the spaces of nonregular generalized functions; in the second subsection we consider a question about an interconnection between Wick calculuses in the regular and nonregular cases; in the third subsection we study an interconnection between the Wick calculus and the operator of stochastic differentiation.

1 PRELIMINARIES

In this paper we denote by $\|\cdot\|_H$ or $|\cdot|_H$ the norm in a space H ; by $(\cdot, \cdot)_H$ the *real*, i.e., *bilinear* scalar product in a space H ; and by $\langle \cdot, \cdot \rangle_H$ or $\langle\langle \cdot, \cdot \rangle\rangle_H$ the dual pairing generated by the scalar product in a space H .

1.1 A Lévy process and its probability space

Denote $\mathbb{R}_+ := [0, +\infty)$. In this paper we deal with a real-valued locally square integrable Lévy process $L = (L_u)_{u \in \mathbb{R}_+}$ (a random process on \mathbb{R}_+ with stationary independent increments and such that $L_0 = 0$) without Gaussian part and drift. As is well known (e.g., [11]), the characteristic function of L is

$$\mathbb{E}[e^{i\theta L_u}] = \exp \left[u \int_{\mathbb{R}} (e^{i\theta x} - 1 - i\theta x) \nu(dx) \right], \quad (1)$$

where ν is the Lévy measure of L , which is a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, here and below \mathcal{B} denotes the Borel σ -algebra; \mathbb{E} denotes the expectation. We assume that ν is a *Radon measure* whose support contains an infinite number of points, $\nu(\{0\}) = 0$, there exists $\varepsilon > 0$ such that $\int_{\mathbb{R}} x^2 e^{\varepsilon|x|} \nu(dx) < \infty$, and $\int_{\mathbb{R}} x^2 \nu(dx) = 1$.

Let us define a measure of the white noise of L . Let \mathcal{D} denote the set of all real-valued infinite-differentiable functions on \mathbb{R}_+ with compact supports. As is well known, \mathcal{D} can be endowed by the projective limit topology generated by a family of Sobolev spaces (e.g., [8]; see also Subsection 1.3). Let \mathcal{D}' be the set of linear continuous functionals on \mathcal{D} . For $\omega \in \mathcal{D}'$ and $\varphi \in \mathcal{D}$ denote $\omega(\varphi)$ by $\langle \omega, \varphi \rangle$; note that actually $\langle \cdot, \cdot \rangle$ is the dual pairing generated by the scalar product in the space $L^2(\mathbb{R}_+)$ of (classes of) square integrable with respect to the Lebesgue measure real-valued functions on \mathbb{R}_+ [8]. The notation $\langle \cdot, \cdot \rangle$ will be preserved for dual pairings in tensor powers of the complexification of a rigging $\mathcal{D}' \supset L^2(\mathbb{R}_+) \supset \mathcal{D}$.

Definition 1. A probability measure μ on $(\mathcal{D}', \mathcal{C}(\mathcal{D}'))$, where \mathcal{C} denotes the cylindrical σ -algebra, with the Fourier transform

$$\int_{\mathcal{D}'} e^{i\langle \omega, \varphi \rangle} \mu(d\omega) = \exp \left[\int_{\mathbb{R}_+ \times \mathbb{R}} (e^{i\varphi(u)x} - 1 - i\varphi(u)x) duv(dx) \right], \quad \varphi \in \mathcal{D}, \quad (2)$$

is called the measure of a Lévy white noise.

The existence of μ follows from the Bochner-Minlos theorem (e.g., [20]), see [38]. Below we assume that the σ -algebra $\mathcal{C}(\mathcal{D}')$ is completed with respect to μ .

Denote by $(L^2) := L^2(\mathcal{D}', \mathcal{C}(\mathcal{D}'), \mu)$ the space of (classes of) complex-valued square integrable with respect to μ functions on \mathcal{D}' (in what follows, this notation will be used very often). Let $f \in L^2(\mathbb{R}_+)$ and a sequence $(\varphi_k \in \mathcal{D})_{k \in \mathbb{N}}$ converge to f in $L^2(\mathbb{R}_+)$ as $k \rightarrow \infty$ (as is well known (e.g., [8]), \mathcal{D} is a dense set in $L^2(\mathbb{R}_+)$). One can show [10, 11, 29, 38] that $\langle \circ, f \rangle := (L^2) - \lim_{k \rightarrow \infty} \langle \circ, \varphi_k \rangle$ is well-defined as an element of (L^2) .

Denote by 1_A the indicator of a set A . Put $1_{[0,0)} \equiv 0$ and consider $\langle \circ, 1_{[0,u]} \rangle \in (L^2)$, $u \in \mathbb{R}_+$. It follows from (1) and (2) that $(\langle \circ, 1_{[0,u]} \rangle)_{u \in \mathbb{R}_+}$ can be identified with a Lévy process on the probability space (triplet) $(\mathcal{D}', \mathcal{C}(\mathcal{D}'), \mu)$ (see [10, 11]). So, one can write $L_u = \langle \circ, 1_{[0,u]} \rangle \in (L^2)$.

Remark 1. The derivative in the sense of generalized functions (e.g., [17]) of a Lévy process (a Lévy white noise) is $L'(\omega) = \langle \omega, \delta \rangle = \omega(\cdot)$, where δ is the Dirac delta-function. Therefore L' is a generalized random process (in the sense of [17]) with trajectories from \mathcal{D}' , and μ is the measure of L' in the classical sense of this notion [18].

Remark 2. A Lévy process L without Gaussian part and drift is a Poisson process if its Lévy measure ν is a point mass at 1, i.e., if for each $\Delta \in \mathcal{B}(\mathbb{R})$ $\nu(\Delta) = \delta_1(\Delta)$. This measure does not satisfy the conditions accepted above (the support of δ_1 does not contain an infinite number of points); nevertheless, all results of the present paper have natural (and often strong) analogs in the Poissonian analysis. The reader can find more information about peculiarities of the Poissonian case in [29], Subsection 1.2.

1.2 Lytvynov's generalization of the CRP

Denote by $\hat{\otimes}$ the symmetric tensor multiplication, by a subscript \mathbb{C} — complexifications of spaces. Set $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. Denote by \mathcal{P} the set of complex-valued polynomials on \mathcal{D}' that consists of zero and elements of the form

$$f(\omega) = \sum_{n=0}^{N_f} \langle \omega^{\hat{\otimes} n}, f^{(n)} \rangle, \quad \omega \in \mathcal{D}', \quad f^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\hat{\otimes} n}, \quad N_f \in \mathbb{Z}_+, \quad f^{(N_f)} \neq 0,$$

here N_f is called the *power of a polynomial* f ; $\langle \omega^{\otimes 0}, f^{(0)} \rangle := f^{(0)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} 0} := \mathbb{C}$. The measure μ of a Lévy white noise has a holomorphic at zero Laplace transform (this follows from (2) and properties of the measure ν , see also [38]), then \mathcal{P} is a dense set in (L^2) [42]. Denote by \mathcal{P}_n , $n \in \mathbb{Z}_+$, the set of polynomials of power smaller or equal to n , by $\overline{\mathcal{P}}_n$ the closure of \mathcal{P}_n in (L^2) . Let for $n \in \mathbb{N}$ $\mathbf{P}_n := \overline{\mathcal{P}}_n \ominus \overline{\mathcal{P}}_{n-1}$ (the orthogonal difference in (L^2)), $\mathbf{P}_0 := \overline{\mathcal{P}}_0$. It is clear that

$$(L^2) = \bigoplus_{n=0}^{\infty} \mathbf{P}_n. \quad (3)$$

Let $f^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n}$, $n \in \mathbb{Z}_+$. Denote by $:\langle \circ^{\otimes n}, f^{(n)} \rangle:$ the orthogonal projection of a monomial $\langle \circ^{\otimes n}, f^{(n)} \rangle$ onto \mathbf{P}_n . Let us define *real*, i.e., *bilinear* scalar products $(\cdot, \cdot)_{ext}$ on $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n}$, $n \in \mathbb{Z}_+$, by setting for $f^{(n)}, g^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n}$

$$(f^{(n)}, g^{(n)})_{ext} := \frac{1}{n!} \int_{\mathcal{D}'} : \langle \omega^{\otimes n}, f^{(n)} \rangle :: \langle \omega^{\otimes n}, g^{(n)} \rangle : \mu(d\omega). \quad (4)$$

The proof of the well-posedness of this definition coincides up to obvious modifications with the proof of the corresponding statement in [38].

By $|\cdot|_{ext}$ we denote the norms corresponding to scalar products (4), i.e.,

$$|f^{(n)}|_{ext} := \sqrt{(f^{(n)}, f^{(n)})_{ext}}.$$

Denote by $\mathcal{H}_{ext}^{(n)}$, $n \in \mathbb{Z}_+$, the completions of $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n}$ with respect to the norms $|\cdot|_{ext}$. For $F^{(n)} \in \mathcal{H}_{ext}^{(n)}$ define a Wick monomial $:\langle \circ^{\otimes n}, F^{(n)} \rangle:$ $\stackrel{\text{def}}{=} (L^2)\text{-}\lim_{k \rightarrow \infty} : \langle \circ^{\otimes n}, f_k^{(n)} \rangle :$, where $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n} \ni f_k^{(n)} \rightarrow F^{(n)}$ as $k \rightarrow \infty$ in $\mathcal{H}_{ext}^{(n)}$ (the well-posedness of this definition can be proved by the method of "mixed sequences"). One can show that $:\langle \circ^{\otimes 0}, F^{(0)} \rangle: = \langle \circ^{\otimes 0}, F^{(0)} \rangle = F^{(0)}$ and $:\langle \circ, F^{(1)} \rangle: = \langle \circ, F^{(1)} \rangle$ (cf. [38]).

Since, as is easy to see, for each $n \in \mathbb{Z}_+$ the set $\{:\langle \circ^{\otimes n}, f^{(n)} \rangle: | f^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n}\}$ is dense in \mathbf{P}_n , the next statement from (3) follows.

Theorem 1. (Lytvynov's generalization of the CRP, cf. [38]) *A random variable $F \in (L^2)$ if and only if there exists a unique sequence of kernels $F^{(n)} \in \mathcal{H}_{ext}^{(n)}$ such that*

$$F = \sum_{n=0}^{\infty} : \langle \circ^{\otimes n}, F^{(n)} \rangle : \quad (5)$$

(the series converges in (L^2)) and $\|F\|_{(L^2)}^2 = \int_{\mathcal{D}'} |F(\omega)|^2 \mu(d\omega) = \mathbb{E}|F|^2 = \sum_{n=0}^{\infty} n! |F^{(n)}|_{ext}^2 < \infty$.

Remark 3. *In order to consider many problems of the Lévy white noise analysis, in terms of Lytvynov's generalization of the CRP, it is necessary to know an explicit formula for the scalar products $(\cdot, \cdot)_{ext}$. Such a formula is calculated in [38]; in another record form (more convenient for some calculations) it is given in, e.g., [13, 15, 16].*

Denote $\mathcal{H} := L^2(\mathbb{R}_+)$, then $\mathcal{H}_{\mathbb{C}} = L^2(\mathbb{R}_+)_{\mathbb{C}}$ (in what follows, this notation will be used very often). It follows from the explicit formula for $(\cdot, \cdot)_{ext}$ that $\mathcal{H}_{ext}^{(1)} = \mathcal{H}_{\mathbb{C}}$, and for $n \in \mathbb{N} \setminus \{1\}$ one can identify $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}$ with the proper subspace of $\mathcal{H}_{ext}^{(n)}$ that consists of "vanishing on diagonals" elements (roughly speaking, such that $F^{(n)}(u_1, \dots, u_n) = 0$ if there exist $k, j \in \{1, \dots, n\}$: $k \neq j$, but $u_k = u_j$). In this sense the space $\mathcal{H}_{ext}^{(n)}$ is an extension of $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}$ (this explains why we use the subscript "ext" in our designations).

1.3 A nonregular rigging of (L^2)

Denote by T the set of indexes $\tau = (\tau_1, \tau_2)$, where $\tau_1 \in \mathbb{N}$, τ_2 is an infinite differentiable function on \mathbb{R}_+ such that for all $u \in \mathbb{R}_+$ $\tau_2(u) \geq 1$. Let \mathcal{H}_τ be the real Sobolev space on \mathbb{R}_+ of order τ_1 weighted by the function τ_2 , i.e., \mathcal{H}_τ is the completion of \mathcal{D} with respect to the norm generated by the scalar product

$$(\varphi, \psi)_{\mathcal{H}_\tau} = \int_{\mathbb{R}_+} \left(\varphi(u)\psi(u) + \sum_{k=1}^{\tau_1} \varphi^{[k]}(u)\psi^{[k]}(u) \right) \tau_2(u) du,$$

here $\varphi^{[k]}$ and $\psi^{[k]}$ are derivatives of order k of functions φ and ψ respectively. It is well known (e.g., [8]) that $\mathcal{D} = \text{pr} \lim_{\tau \in T} \mathcal{H}_\tau$ (moreover, for any $n \in \mathbb{N}$ $\mathcal{D}^{\widehat{\otimes} n} = \text{pr} \lim_{\tau \in T} \mathcal{H}_\tau^{\widehat{\otimes} n}$, see, e.g., [6] for details), and for each $\tau \in T$ \mathcal{H}_τ is densely and continuously embedded into $\mathcal{H} \equiv L^2(\mathbb{R}_+)$. Therefore one can consider the chain

$$\mathcal{D}' \supset \mathcal{H}_{-\tau} \supset \mathcal{H} \supset \mathcal{H}_\tau \supset \mathcal{D},$$

where $\mathcal{H}_{-\tau}$, $\tau \in T$, are the spaces dual of \mathcal{H}_τ with respect to \mathcal{H} . Note that by the Schwartz theorem [8] $\mathcal{D}' = \text{ind} \lim_{\tau \in T} \mathcal{H}_{-\tau}$ (it is convenient for us to consider \mathcal{D}' as a topological space with the inductive limit topology). By analogy with [28] one can easily show that the measure μ of a Lévy white noise is concentrated on $\mathcal{H}_{-\tilde{\tau}}$ with some $\tilde{\tau} \in T$, i.e., $\mu(\mathcal{H}_{-\tilde{\tau}}) = 1$. Excepting from T the indexes τ such that μ is not concentrated on $\mathcal{H}_{-\tau}$, we will assume, in what follows, that for each $\tau \in T$ $\mu(\mathcal{H}_{-\tau}) = 1$.

Denote the norms in $\mathcal{H}_{\tau, \mathbb{C}}$ and its tensor powers by $|\cdot|_\tau$, i.e., for $f^{(n)} \in \mathcal{H}_{\tau, \mathbb{C}}^{\widehat{\otimes} n}$, $n \in \mathbb{Z}_+$, $|f^{(n)}|_\tau = \sqrt{(f^{(n)}, \overline{f^{(n)}})}_{\mathcal{H}_{\tau, \mathbb{C}}^{\widehat{\otimes} n}}$ (note that $\mathcal{H}_{\tau, \mathbb{C}}^{\widehat{\otimes} 0} := \mathbb{C}$ and $|f^{(0)}|_\tau = |f^{(0)}|$).

The next statement easily follows from results of [25].

Lemma 1. *There exists $\tau' \in T$ such that for each $n \in \mathbb{Z}_+$ the space $\mathcal{H}_{\tau', \mathbb{C}}^{\widehat{\otimes} n}$ is densely and continuously embedded into the space $\mathcal{H}_{\text{ext}}^{(n)}$. Moreover, for all $f^{(n)} \in \mathcal{H}_{\tau', \mathbb{C}}^{\widehat{\otimes} n}$ $|f^{(n)}|_{\text{ext}}^2 \leq n!c^n |f^{(n)}|_{\tau'}^2$, where $c > 0$ is some constant.*

It follows from this lemma that if for some $\tau \in T$ the space \mathcal{H}_τ is continuously embedded into the space $\mathcal{H}_{\tau'}$, then for each $n \in \mathbb{Z}_+$ the space $\mathcal{H}_{\tau, \mathbb{C}}^{\widehat{\otimes} n}$ is densely and continuously embedded into the space $\mathcal{H}_{\text{ext}}^{(n)}$, and there exists $c(\tau) > 0$ such that for all $f^{(n)} \in \mathcal{H}_{\tau, \mathbb{C}}^{\widehat{\otimes} n}$

$$|f^{(n)}|_{\text{ext}}^2 \leq n!c(\tau)^n |f^{(n)}|_\tau^2. \quad (6)$$

In what follows, it will be convenient to assume that the indexes τ such that \mathcal{H}_τ is not continuously embedded into $\mathcal{H}_{\tau'}$, are removed from T .

Accept on default $q \in \mathbb{Z}_+$ and $\tau \in T$. Denote $\mathcal{P}_W := \{f = \sum_{n=0}^{N_f} \langle \circ^{\otimes n}, f^{(n)} \rangle, f^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} n}, N_f \in \mathbb{Z}_+\} \subset (L^2)$. Define real scalar products $(\cdot, \cdot)_{\tau, q}$ on \mathcal{P}_W by setting for

$$\begin{aligned} f &= \sum_{n=0}^{N_f} \langle \circ^{\otimes n}, f^{(n)} \rangle, \quad g = \sum_{n=0}^{N_g} \langle \circ^{\otimes n}, g^{(n)} \rangle \in \mathcal{P}_W \\ (f, g)_{\tau, q} &:= \sum_{n=0}^{\min(N_f, N_g)} (n!)^2 2^{qn} (f^{(n)}, g^{(n)})_{\mathcal{H}_{\tau, \mathbb{C}}^{\widehat{\otimes} n}}. \end{aligned} \quad (7)$$

Let $\|\cdot\|_{\tau,q}$ be the corresponding norms, i.e., $\|f\|_{\tau,q} = \sqrt{(f, \bar{f})_{\tau,q}}$. In order to verify the well-posedness of this definition, i.e., that formula (7) defines *scalar*, and not just quasiscalar products, we note that if $f \in \mathcal{P}_W$ and $\|f\|_{\tau,q} = 0$ then by (7) for each kernel $f^{(n)}$ we have $|f^{(n)}|_{\tau} = 0$ and therefore by (6) $|f^{(n)}|_{ext} = 0$. So, in this case $f = 0$ in (L^2) .

Let $(\mathcal{H}_{\tau})_q$ be completions of \mathcal{P}_W with respect to the norms $\|\cdot\|_{\tau,q}$, $(\mathcal{H}_{\tau}) := \text{pr lim}_{q \rightarrow \infty} (\mathcal{H}_{\tau})_q$, $(\mathcal{D}) := \text{pr lim}_{\tau \in T, q \rightarrow \infty} (\mathcal{H}_{\tau})_q$. As is easy to see, $f \in (\mathcal{H}_{\tau})_q$ if and only if f can be presented in the form

$$f = \sum_{n=0}^{\infty} : \langle \circ^{\otimes n}, f^{(n)} \rangle :, f^{(n)} \in \mathcal{H}_{\tau, \mathbb{C}}^{\widehat{\otimes} n} \quad (8)$$

(the series converges in $(\mathcal{H}_{\tau})_q$), with

$$\|f\|_{\tau,q}^2 := \|f\|_{(\mathcal{H}_{\tau})_q}^2 = \sum_{n=0}^{\infty} (n!)^2 2^{qn} |f^{(n)}|_{\tau}^2 < \infty \quad (9)$$

(since for each $n \in \mathbb{Z}_+$ $\mathcal{H}_{\tau, \mathbb{C}}^{\widehat{\otimes} n} \subseteq \mathcal{H}_{ext}^{(n)}$, for $f^{(n)} \in \mathcal{H}_{\tau, \mathbb{C}}^{\widehat{\otimes} n} : \langle \circ^{\otimes n}, f^{(n)} \rangle :$ is a well defined Wick monomial, see Subsection 1.2). Further, $f \in (\mathcal{H}_{\tau})$ ($f \in (\mathcal{D})$) if and only if f can be presented in form (8) and norm (9) is finite for each $q \in \mathbb{Z}_+$ (for each $\tau \in T$ and each $q \in \mathbb{Z}_+$).

Lemma 2. *For each $\tau \in T$ there exists $q_0(\tau) \in \mathbb{Z}_+$ such that the space $(\mathcal{H}_{\tau})_q$ is densely and continuously embedded into (L^2) for each $q \in \mathbb{N}_{q_0(\tau)} := \{q_0(\tau), q_0(\tau) + 1, \dots\}$.*

The proof coincides up to obvious modifications with the proof of the corresponding statement in the real case [25]. In view of this lemma one can consider a chain

$$(\mathcal{D}') \supset (\mathcal{H}_{-\tau}) \supset (\mathcal{H}_{-\tau})_{-q} \supset (L^2) \supset (\mathcal{H}_{\tau})_q \supset (\mathcal{H}_{\tau}) \supset (\mathcal{D}), \quad \tau \in T, \quad q \in \mathbb{N}_{q_0(\tau)}, \quad (10)$$

where $(\mathcal{H}_{-\tau})_{-q}$, $(\mathcal{H}_{-\tau}) = \text{ind lim}_{q' \rightarrow \infty} (\mathcal{H}_{-\tau})_{-q'}$ and $(\mathcal{D}') = \text{ind lim}_{\hat{\tau} \in T, q' \rightarrow \infty} (\mathcal{H}_{-\hat{\tau}})_{-q'}$ are the spaces dual of $(\mathcal{H}_{\tau})_q$, (\mathcal{H}_{τ}) and (\mathcal{D}) with respect to (L^2) .

Definition 2. *Chain (10) is called a nonregular rigging of the space (L^2) . The positive spaces of this chain $(\mathcal{H}_{\tau})_q$, (\mathcal{H}_{τ}) and (\mathcal{D}) are called Kondratiev spaces of nonregular test functions. The negative spaces of this chain $(\mathcal{H}_{-\tau})_{-q}$, $(\mathcal{H}_{-\tau})$ and (\mathcal{D}') are called Kondratiev spaces of nonregular generalized functions.*

Finally, we describe natural orthogonal bases in the spaces $(\mathcal{H}_{-\tau})_{-q}$. Let us consider chains

$$\mathcal{D}_{\mathbb{C}}^{(m)} \supset \mathcal{H}_{-\tau, \mathbb{C}}^{(m)} \supset \mathcal{H}_{ext}^{(m)} \supset \mathcal{H}_{\tau, \mathbb{C}}^{\widehat{\otimes} m} \supset \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} m}, \quad (11)$$

$m \in \mathbb{Z}_+$ (for $m = 0$ $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} 0} = \mathcal{H}_{\tau, \mathbb{C}}^{\widehat{\otimes} 0} = \mathcal{H}_{ext}^{(0)} = \mathcal{H}_{-\tau, \mathbb{C}}^{(0)} = \mathcal{D}_{\mathbb{C}}^{(0)} = \mathbb{C}$), where $\mathcal{H}_{-\tau, \mathbb{C}}^{(m)}$ and $\mathcal{D}_{\mathbb{C}}^{(m)} = \text{ind lim}_{\hat{\tau} \in T} \mathcal{H}_{-\hat{\tau}, \mathbb{C}}^{(m)}$ are the spaces dual of $\mathcal{H}_{\tau, \mathbb{C}}^{\widehat{\otimes} m}$ and $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} m}$ with respect to $\mathcal{H}_{ext}^{(m)}$. In what follows, we denote by $\langle \cdot, \cdot \rangle_{ext}$ the real dual pairings between elements of negative and positive spaces from chains (11), these pairings are generated by the scalar products in $\mathcal{H}_{ext}^{(m)}$. The next statement follows from the definition of the spaces $(\mathcal{H}_{-\tau})_{-q}$ and the general duality theory (cf. [25, 28]).

Proposition 1. *There exists a system of generalized functions*

$$\{:\langle \circ^{\otimes m}, F_{ext}^{(m)} \rangle : \in (\mathcal{H}_{-\tau})_{-q} \mid F_{ext}^{(m)} \in \mathcal{H}_{-\tau, \mathbb{C}}^{(m)}, m \in \mathbb{Z}_+\}$$

such that

1) for $F_{ext}^{(m)} \in \mathcal{H}_{ext}^{(m)} \subset \mathcal{H}_{-\tau, \mathbb{C}}^{(m)} : \langle \circ^{\otimes m}, F_{ext}^{(m)} \rangle :$ is a Wick monomial that is defined in Subsection 1.2;

2) any generalized function $F \in (\mathcal{H}_{-\tau})_{-q}$ can be presented as a series

$$F = \sum_{m=0}^{\infty} : \langle \circ^{\otimes m}, F_{ext}^{(m)} \rangle :, F_{ext}^{(m)} \in \mathcal{H}_{-\tau, \mathbb{C}}^{(m)} \quad (12)$$

that converges in $(\mathcal{H}_{-\tau})_{-q}$, i.e.,

$$\|F\|_{-\tau, -q}^2 := \|F\|_{(\mathcal{H}_{-\tau})_{-q}}^2 = \sum_{m=0}^{\infty} 2^{-qm} |F_{ext}^{(m)}|_{\mathcal{H}_{-\tau, \mathbb{C}}^{(m)}}^2 < \infty; \quad (13)$$

and, vice versa, any series (12) with finite norm (13) is a generalized function from $(\mathcal{H}_{-\tau})_{-q}$ (i.e., such a series converges in $(\mathcal{H}_{-\tau})_{-q}$);

3) the dual pairing between $F \in (\mathcal{H}_{-\tau})_{-q}$ and $f \in (\mathcal{H}_{\tau})_q$ that is generated by the scalar product in (L^2) , has the form

$$\langle\langle F, f \rangle\rangle_{(L^2)} = \sum_{m=0}^{\infty} m! \langle F_{ext}^{(m)}, f^{(m)} \rangle_{ext}, \quad (14)$$

where $F_{ext}^{(m)} \in \mathcal{H}_{-\tau, \mathbb{C}}^{(m)}$ and $f^{(m)} \in \mathcal{H}_{\tau, \mathbb{C}}^{\widehat{\otimes} m}$ are the kernels from decompositions (12) and (8) for F and f respectively.

It is clear that $F \in (\mathcal{H}_{-\tau})$ ($F \in (\mathcal{D}')$) if and only if F can be presented in form (12) and norm (13) is finite for some $q \in \mathbb{N}_{q_0(\tau)}$ (for some $\tau \in T$ and some $q \in \mathbb{N}_{q_0(\tau)}$).

2 ELEMENTS OF A WICK CALCULUS

In this paper we construct a Wick calculus on the spaces $(\mathcal{H}_{-\tau})$; but, as is easy to verify, all our results hold true up to obvious modifications on the space (\mathcal{D}') .

2.1 A Wick product and Wick versions of holomorphic functions

One can introduce a Wick product and Wick versions of holomorphic functions on $(\mathcal{H}_{-\tau})$ by different ways. We use the most natural and convenient from technical point of view classical way, based on a so-called S -transform.

Definition 3. *Let $F \in (\mathcal{H}_{-\tau})$. We define an S -transform $(SF)(\lambda)$, $\lambda \in \mathcal{D}_{\mathbb{C}}$, as a formal series*

$$(SF)(\lambda) := \sum_{m=0}^{\infty} \langle F_{ext}^{(m)}, \lambda^{\otimes m} \rangle_{ext} \equiv F_{ext}^{(0)} + \sum_{m=1}^{\infty} \langle F_{ext}^{(m)}, \lambda^{\otimes m} \rangle_{ext}, \quad (15)$$

where $F_{ext}^{(m)} \in \mathcal{H}_{-\tau, \mathbb{C}}^{(m)}$ are the kernels from (12) for F . In particular, $(SF)(0) = F_{ext}^{(0)}$, $S1 \equiv 1$.

Remark 4. As is easily seen, each term in series (15) is well-defined, but the series can diverge. However, the last is not an obstruction in order to construct the Wick calculus (cf. [15]); moreover, it is easy to obtain a simple sufficient condition under which series (15) converges. Namely, by the generalized and classical Cauchy-Bunyakovsky inequalities

$$\begin{aligned} |(SF)(\lambda)| &\leq \sum_{m=0}^{\infty} |F_{ext}^{(m)}|_{\mathcal{H}_{-\tau, \mathbb{C}}^{(m)}} |\lambda|_{\tau}^m = \sum_{m=0}^{\infty} (2^{-qm/2} |F_{ext}^{(m)}|_{\mathcal{H}_{-\tau, \mathbb{C}}^{(m)}}) (2^{qm/2} |\lambda|_{\tau}^m) \\ &\leq \sqrt{\sum_{m=0}^{\infty} 2^{-qm} |F_{ext}^{(m)}|_{\mathcal{H}_{-\tau, \mathbb{C}}^{(m)}}^2} \sqrt{\sum_{m=0}^{\infty} 2^{qm} |\lambda|_{\tau}^{2m}} = \|F\|_{-\tau, -q} \sqrt{\sum_{m=0}^{\infty} 2^{qm} |\lambda|_{\tau}^{2m}} \end{aligned}$$

(see (13)). Therefore series (15) converges if $F \in (\mathcal{H}_{-\tau})_{-q}$ and $\lambda \in \mathcal{D}_{\mathbb{C}}$ is such that $|\lambda|_{\tau} < 2^{-q/2}$. Note that the last inequality is true if and only if a function $f_{\lambda}(\circ) := \sum_{m=0}^{\infty} \frac{1}{m!} \langle \circ^{\otimes m}, \lambda^{\otimes m} \rangle : \in (\mathcal{H}_{\tau})_q$, in this case $\|f_{\lambda}\|_{\tau, q} = \sqrt{\sum_{m=0}^{\infty} 2^{qm} |\lambda|_{\tau}^{2m}} < \infty$ (see (9)). Now one can define the S -transform of F by the formula $(SF)(\lambda) = \langle F, f_{\lambda} \rangle_{(L^2)}$ (cf. [34]), see (14). Note that in the Gaussian (and Poissonian) analysis $f_{\lambda}(\circ) = \exp^{\diamond} \{ \langle \circ, \lambda \rangle \}$, where \exp^{\diamond} is a Wick version of the exponential function (e.g., [34]), and therefore f_{λ} is called a Wick exponential; in the Lévy analysis this representation for f_{λ} does not hold.

Definition 4. For $F, G \in (\mathcal{H}_{-\tau})$ and a holomorphic at $(SF)(0)$ function $h : \mathbb{C} \rightarrow \mathbb{C}$ we define a Wick product $F \diamond G$ and a Wick version $h^{\diamond}(F)$ by setting formally

$$F \diamond G := S^{-1}(SF \cdot SG), \quad h^{\diamond}(F) := S^{-1}h(SF). \quad (16)$$

It is obvious that the Wick multiplication \diamond is commutative, associative and distributive over a field \mathbb{C} .

Remark 5. A function h can be decomposed in a Taylor series

$$h(u) = \sum_{m=0}^{\infty} h_m (u - (SF)(0))^m. \quad (17)$$

Using this decomposition, it is easy to calculate that

$$h^{\diamond}(F) = \sum_{m=0}^{\infty} h_m (F - (SF)(0))^{\diamond m}, \quad (18)$$

where $F^{\diamond m} := \underbrace{F \diamond \dots \diamond F}_{m \text{ times}}, F^{\diamond 0} := 1$.

Let us write out "coordinate formulas" for the Wick product and for the Wick versions of holomorphic functions (i.e., representations of $F \diamond G$ and $h^{\diamond}(F)$ via kernels from decompositions (12) for F and G and coefficients from decomposition (17) for h). We need a small preparation: it is necessary to introduce an analog of the symmetric tensor multiplication on the spaces $\mathcal{H}_{-\tau, \mathbb{C}}^{(m)}, m \in \mathbb{Z}_+$.

Consider a family of chains

$$\mathcal{D}'_{\mathbb{C}} \hat{\otimes}^m \supset \mathcal{H}_{-\tau, \mathbb{C}} \hat{\otimes}^m \supset \mathcal{H}_{\mathbb{C}} \hat{\otimes}^m \supset \mathcal{H}_{\tau, \mathbb{C}} \hat{\otimes}^m \supset \mathcal{D}_{\mathbb{C}} \hat{\otimes}^m, \quad m \in \mathbb{Z}_+ \quad (19)$$

(as is well known (e.g., [6, 8]), $\mathcal{H}_{-\tau, \mathbb{C}}^{\widehat{\otimes} m}$ and $\mathcal{D}'_{\mathbb{C}}^{\widehat{\otimes} m} = \text{ind} \lim_{\widehat{\tau} \in T} \mathcal{H}_{-\widehat{\tau}, \mathbb{C}}^{\widehat{\otimes} m}$ are the spaces dual of $\mathcal{H}_{\tau, \mathbb{C}}^{\widehat{\otimes} m}$ and $\mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} m}$ respectively; in the case $m = 0$ all spaces from chain (19) are equal to \mathbb{C}). Since the spaces of test functions in chains (19) and (11) coincide, there exists a family of natural isomorphisms $U_m : \mathcal{D}'_{\mathbb{C}}^{(m)} \rightarrow \mathcal{D}'_{\mathbb{C}}^{\widehat{\otimes} m}$, $m \in \mathbb{Z}_+$, such that for all $F_{ext}^{(m)} \in \mathcal{D}'_{\mathbb{C}}^{(m)}$ and $f^{(m)} \in \mathcal{D}_{\mathbb{C}}^{\widehat{\otimes} m}$

$$\langle F_{ext}^{(m)}, f^{(m)} \rangle_{ext} = \langle U_m F_{ext}^{(m)}, f^{(m)} \rangle. \quad (20)$$

It is easy to see that the restrictions of U_m to $\mathcal{H}_{-\tau, \mathbb{C}}^{(m)}$ are isometric isomorphisms between the spaces $\mathcal{H}_{-\tau, \mathbb{C}}^{(m)}$ and $\mathcal{H}_{-\tau, \mathbb{C}}^{\widehat{\otimes} m}$.

Remark 6. As we saw above, $\mathcal{H}_{ext}^{(1)} = \mathcal{H}_{\mathbb{C}}$, therefore in the case $m = 1$ chains (19) and (11) coincide. Thus U_1 is the identity operator on $\mathcal{D}'_{\mathbb{C}}^{(1)} = \mathcal{D}'_{\mathbb{C}}^{\widehat{\otimes} 1} = \mathcal{D}'_{\mathbb{C}}$. In the case $m = 0$ U_0 is, obviously, the identity operator on \mathbb{C} .

For $F_{ext}^{(n)} \in \mathcal{H}_{-\tau, \mathbb{C}}^{(n)}$ and $G_{ext}^{(m)} \in \mathcal{H}_{-\tau, \mathbb{C}}^{(m)}$, $n, m \in \mathbb{Z}_+$, set

$$F_{ext}^{(n)} \diamond G_{ext}^{(m)} := U_{n+m}^{-1} [(U_n F_{ext}^{(n)}) \widehat{\otimes} (U_m G_{ext}^{(m)})] \in \mathcal{H}_{-\tau, \mathbb{C}}^{(n+m)}. \quad (21)$$

It follows from properties of operators U_m and of the symmetric tensor multiplication that the multiplication \diamond is commutative, associative and distributive over a field \mathbb{C} . Further, since $U_m : \mathcal{H}_{-\tau, \mathbb{C}}^{(m)} \rightarrow \mathcal{H}_{-\tau, \mathbb{C}}^{\widehat{\otimes} m}$, $m \in \mathbb{Z}_+$, are isometric isomorphisms,

$$\begin{aligned} |F_{ext}^{(n)} \diamond G_{ext}^{(m)}|_{\mathcal{H}_{-\tau, \mathbb{C}}^{(n+m)}} &= |(U_n F_{ext}^{(n)}) \widehat{\otimes} (U_m G_{ext}^{(m)})|_{\mathcal{H}_{-\tau, \mathbb{C}}^{\widehat{\otimes} n+m}} \\ &\leq |U_n F_{ext}^{(n)}|_{\mathcal{H}_{-\tau, \mathbb{C}}^{\widehat{\otimes} n}} |U_m G_{ext}^{(m)}|_{\mathcal{H}_{-\tau, \mathbb{C}}^{\widehat{\otimes} m}} = |F_{ext}^{(n)}|_{\mathcal{H}_{-\tau, \mathbb{C}}^{(n)}} |G_{ext}^{(m)}|_{\mathcal{H}_{-\tau, \mathbb{C}}^{(m)}}. \end{aligned} \quad (22)$$

Finally, by (20) and (21) for $\lambda \in \mathcal{D}_{\mathbb{C}}$

$$\begin{aligned} \langle F_{ext}^{(n)}, \lambda^{\otimes n} \rangle_{ext} \langle G_{ext}^{(m)}, \lambda^{\otimes m} \rangle_{ext} &= \langle U_n F_{ext}^{(n)}, \lambda^{\otimes n} \rangle \langle U_m G_{ext}^{(m)}, \lambda^{\otimes m} \rangle \\ &= \langle (U_n F_{ext}^{(n)}) \otimes (U_m G_{ext}^{(m)}), \lambda^{\otimes n+m} \rangle = \langle (U_n F_{ext}^{(n)}) \widehat{\otimes} (U_m G_{ext}^{(m)}), \lambda^{\otimes n+m} \rangle \\ &= \langle U_{n+m}^{-1} [(U_n F_{ext}^{(n)}) \widehat{\otimes} (U_m G_{ext}^{(m)})], \lambda^{\otimes n+m} \rangle_{ext} = \langle F_{ext}^{(n)} \diamond G_{ext}^{(m)}, \lambda^{\otimes n+m} \rangle_{ext}. \end{aligned}$$

Using (16), (15) and this equality, by analogy with the Meixner analysis [28] one can prove the following statement.

Proposition 2. For $F_1, \dots, F_n \in (\mathcal{H}_{-\tau})$

$$F_1 \diamond \dots \diamond F_n = \sum_{m=0}^{\infty} : \langle \circ^{\otimes m}, \sum_{\substack{k_1, \dots, k_n \in \mathbb{Z}_+ \\ k_1 + \dots + k_n = m}} F_1^{(k_1)} \diamond \dots \diamond F_n^{(k_n)} \rangle :; \quad (23)$$

in particular, for $F, G \in (\mathcal{H}_{-\tau})$

$$F \diamond G = \sum_{m=0}^{\infty} : \langle \circ^{\otimes m}, \sum_{k=0}^m F_{ext}^{(k)} \diamond G_{ext}^{(m-k)} \rangle :. \quad (24)$$

Here $F_j^{(k_j)} \in \mathcal{H}_{-\tau, \mathbb{C}}^{(k_j)}$, $j \in \{1, \dots, n\}$, are the kernels from decompositions (12) for F_j ; $F_{ext}^{(k)} \in \mathcal{H}_{-\tau, \mathbb{C}}^{(k)}$, $G_{ext}^{(m-k)} \in \mathcal{H}_{-\tau, \mathbb{C}}^{(m-k)}$, are the kernels from the same decompositions for F and G respectively. Further, for $F \in (\mathcal{H}_{-\tau})$ and a holomorphic at $(SF)(0) = F_{ext}^{(0)}$ function $h : \mathbb{C} \rightarrow \mathbb{C}$

$$h^\diamond(F) = h_0 + \sum_{m=1}^{\infty} : \langle \circ^{\otimes m}, \sum_{n=1}^m h_n \sum_{\substack{k_1, \dots, k_n \in \mathbb{N}: \\ k_1 + \dots + k_n = m}} F_{ext}^{(k_1)} \diamond \dots \diamond F_{ext}^{(k_n)} \rangle :, \quad (25)$$

where $F_{ext}^{(k)} \in \mathcal{H}_{-\tau, \mathbb{C}}^{(k)}$, $k \in \mathbb{Z}_+$, are the kernels from decomposition (12) for F ; $h_n \in \mathbb{C}$, $n \in \mathbb{Z}_+$, are the coefficients from decomposition (17) for h .

Remark 7. Formulas (24) and (25) can be used as alternative definitions of the Wick product and of the Wick version of a holomorphic function respectively.

It is clear that in order to give an informal sense to notions "the Wick product" and "the Wick version of a holomorphic function", it is necessary to study a question about convergence of series (23) and (25) in the spaces $(\mathcal{H}_{-\tau})$.

Theorem 2. 1) Let $F_1, \dots, F_n \in (\mathcal{H}_{-\tau})$. Then $F_1 \diamond \dots \diamond F_n \in (\mathcal{H}_{-\tau})$. Moreover, the Wick multiplication is continuous in the sense that

$$\|F_1 \diamond \dots \diamond F_n\|_{-\tau, -q} \leq \sqrt{\max_{m \in \mathbb{Z}_+} [2^{-m} (m+1)^{n-1}]} \|F_1\|_{-\tau, -(q-1)} \dots \|F_n\|_{-\tau, -(q-1)}, \quad (26)$$

where $q \in \mathbb{N}$ is such that $F_1, \dots, F_n \in (\mathcal{H}_{-\tau})_{-(q-1)}$. 2) Let $F \in (\mathcal{H}_{-\tau})$ and a function $h : \mathbb{C} \rightarrow \mathbb{C}$ be holomorphic at $(SF)(0)$. Then $h^\diamond(F) \in (\mathcal{H}_{-\tau})$.

Proof. 1) It is sufficient to prove (26), the fact that $F_1 \diamond \dots \diamond F_n \in (\mathcal{H}_{-\tau})$ follows from this estimate. Let $F_j^{(k)} \in \mathcal{H}_{-\tau, \mathbb{C}}^{(k)}$ be the kernels from decompositions (12) for F_j , $j \in \{1, \dots, n\}$; and $q \in \mathbb{N}$ be such that $F_1, \dots, F_n \in (\mathcal{H}_{-\tau})_{-(q-1)}$ (such q exists because by Schwartz's theorem $(\mathcal{H}_{-\tau}) = \bigcup_{q \in \mathbb{N}_{q_0(\tau)}} (\mathcal{H}_{-\tau})_{-q}$ ($\mathbb{N}_{q_0(\tau)}$ is defined in Lemma 2), see, e.g., [8] for details). Using (23),

(13), a known estimate for a norm $\|\sum_{l=1}^p a_l\|^2 \leq p \sum_{l=1}^p \|a_l\|^2$ and (22), we obtain

$$\begin{aligned} \|F_1 \diamond \dots \diamond F_n\|_{-\tau, -q}^2 &= \sum_{m=0}^{\infty} 2^{-qm} \left| \sum_{\substack{k_1, \dots, k_n \in \mathbb{Z}_+ : \\ k_1 + \dots + k_n = m}} F_1^{(k_1)} \diamond \dots \diamond F_n^{(k_n)} \right|_{\mathcal{H}_{-\tau, \mathbb{C}}^{(m)}}^2 \\ &= \sum_{m=0}^{\infty} 2^{-qm} \left| \sum_{k_1=0}^m \sum_{k_2=0}^{m-k_1} \dots \sum_{k_{n-1}=0}^{m-k_1-\dots-k_{n-2}} F_1^{(k_1)} \diamond \dots \diamond F_{n-1}^{(k_{n-1})} \diamond F_n^{(m-k_1-\dots-k_{n-1})} \right|_{\mathcal{H}_{-\tau, \mathbb{C}}^{(m)}}^2 \\ &\leq \sum_{m=0}^{\infty} 2^{-qm} (m+1) \sum_{k_1=0}^m \left| \sum_{k_2=0}^{m-k_1} \dots \sum_{k_{n-1}=0}^{m-k_1-\dots-k_{n-2}} F_1^{(k_1)} \diamond \dots \diamond F_n^{(m-k_1-\dots-k_{n-1})} \right|_{\mathcal{H}_{-\tau, \mathbb{C}}^{(m)}}^2 \\ &\leq \dots \leq \sum_{m=0}^{\infty} 2^{-qm} (m+1)^{n-1} \sum_{k_1=0}^m \sum_{k_2=0}^{m-k_1} \dots \sum_{k_{n-1}=0}^{m-k_1-\dots-k_{n-2}} |F_1^{(k_1)} \diamond \dots \diamond F_n^{(m-k_1-\dots-k_{n-1})}|_{\mathcal{H}_{-\tau, \mathbb{C}}^{(m)}}^2 \\ &\leq \sum_{m=0}^{\infty} [2^{-m} (m+1)^{n-1}] 2^{-(q-1)m} \end{aligned}$$

$$\begin{aligned}
 & \times \sum_{k_1=0}^m \sum_{k_2=0}^{m-k_1} \cdots \sum_{k_{n-1}=0}^{m-k_1-\cdots-k_{n-2}} |F_1^{(k_1)}|_{\mathcal{H}_{-\tau,\mathbb{C}}^{(k_1)}}^2 \cdots |F_n^{(m-k_1-\cdots-k_{n-1})}|_{\mathcal{H}_{-\tau,\mathbb{C}}^{(m-k_1-\cdots-k_{n-1})}}^2 \\
 & \leq C(n) \sum_{k_1=0}^{\infty} 2^{-(q-1)k_1} |F_1^{(k_1)}|_{\mathcal{H}_{-\tau,\mathbb{C}}^{(k_1)}}^2 \sum_{m=k_1}^{\infty} \sum_{k_2=0}^{m-k_1} \cdots \sum_{k_{n-1}=0}^{m-k_1-\cdots-k_{n-2}} 2^{-(q-1)k_2} |F_2^{(k_2)}|_{\mathcal{H}_{-\tau,\mathbb{C}}^{(k_2)}}^2 \\
 & \quad \cdots 2^{-(q-1)(m-k_1-\cdots-k_{n-1})} |F_n^{(m-k_1-\cdots-k_{n-1})}|_{\mathcal{H}_{-\tau,\mathbb{C}}^{(m-k_1-\cdots-k_{n-1})}}^2 \\
 & = C(n) \|F_1\|_{-\tau, -(q-1)}^2 \sum_{m=0}^{\infty} \sum_{k_2=0}^m \cdots \sum_{k_{n-1}=0}^{m-k_2-\cdots-k_{n-2}} 2^{-(q-1)k_2} |F_2^{(k_2)}|_{\mathcal{H}_{-\tau,\mathbb{C}}^{(k_2)}}^2 \\
 & \quad \cdots 2^{-(q-1)(m-k_2-\cdots-k_{n-1})} |F_n^{(m-k_2-\cdots-k_{n-1})}|_{\mathcal{H}_{-\tau,\mathbb{C}}^{(m-k_2-\cdots-k_{n-1})}}^2 \\
 & = \cdots = C(n) \|F_1\|_{-\tau, -(q-1)}^2 \cdots \|F_n\|_{-\tau, -(q-1)}^2, \tag{27}
 \end{aligned}$$

where $C(n) := \max_{m \in \mathbb{Z}_+} [2^{-m}(m+1)^{n-1}]$.

2) Let us establish that for some $q \in \mathbb{Z}_+$ $\|h^\diamond(F)\|_{-\tau, -q} < \infty$, it is enough to assert that $h^\diamond(F) \in (\mathcal{H}_{-\tau})$. Let $F_{ext}^{(k)} \in \mathcal{H}_{-\tau,\mathbb{C}}^{(k)}$, $k \in \mathbb{Z}_+$, be the kernels from decomposition (12) for F . Since by Schwartz's theorem for some $\tilde{q} \in \mathbb{Z}_+$ $F \in (\mathcal{H}_{-\tau})_{-\tilde{q}}$, by (13) for each k we have $|F_{ext}^{(k)}|_{\mathcal{H}_{-\tau,\mathbb{C}}^{(k)}} \leq \|F\|_{-\tau, -\tilde{q}} 2^{\tilde{q}k/2}$. Further, it follows from the holomorphy of h that there exists $q' \in \mathbb{Z}_+$ such that for each $n \in \mathbb{Z}_+$ $|h_n| \leq 2^{q'n}$, where $h_n \in \mathbb{C}$ are the coefficients from decomposition (17) for h . Using these estimates, (13), (25), (22) and the estimate $\sum_{\substack{k_1, \dots, k_n \in \mathbb{N}: \\ k_1 + \dots + k_n = m}} 1 = C_{m-1}^{n-1} \leq 2^{m-1}$, we obtain

$$\begin{aligned}
 \|h^\diamond(F)\|_{-\tau, -q}^2 & = |h_0|^2 + \sum_{m=1}^{\infty} 2^{-qm} \left| \sum_{n=1}^m h_n \sum_{\substack{k_1, \dots, k_n \in \mathbb{N}: \\ k_1 + \dots + k_n = m}} F_{ext}^{(k_1)} \diamond \cdots \diamond F_{ext}^{(k_n)} \right|_{\mathcal{H}_{-\tau,\mathbb{C}}^{(m)}}^2 \\
 & \leq |h_0|^2 + \sum_{m=1}^{\infty} 2^{-qm} \left(\sum_{n=1}^m |h_n| \sum_{\substack{k_1, \dots, k_n \in \mathbb{N}: \\ k_1 + \dots + k_n = m}} |F_{ext}^{(k_1)}|_{\mathcal{H}_{-\tau,\mathbb{C}}^{(k_1)}} \cdots |F_{ext}^{(k_n)}|_{\mathcal{H}_{-\tau,\mathbb{C}}^{(k_n)}} \right)^2 \\
 & \leq |h_0|^2 + \sum_{m=1}^{\infty} 2^{-qm} \left(\sum_{n=1}^m 2^{q'n} \sum_{\substack{k_1, \dots, k_n \in \mathbb{N}: \\ k_1 + \dots + k_n = m}} \|F\|_{-\tau, -\tilde{q}}^{2n} 2^{\tilde{q}m/2} \right)^2 \\
 & \leq |h_0|^2 + \frac{1}{4} \sum_{m=1}^{\infty} 2^{(\tilde{q}+2-q)m} \left(\sum_{n=1}^m (2^{q'} \|F\|_{-\tau, -\tilde{q}})^n \right)^2 < \infty, \tag{28}
 \end{aligned}$$

if $q \in \mathbb{Z}_+$ is sufficiently large. □

Remark 8. Let $h_N^\diamond(F)$, $N \in \mathbb{N}$, be the Wick version of the N -th partial sum of decomposition (17) for h . It follows from calculation (28) that $h_N^\diamond(F) \rightarrow h^\diamond(F)$ as $N \rightarrow \infty$ in $(\mathcal{H}_{-\tau})$.

Remark 9. One of generalizations of the Gaussian white noise analysis is a so-called biorthogonal analysis (see [1,2,5,23,24,36]) that developed actively in 90th of the last century. Its main idea is to use as orthogonal bases in spaces of test functions so-called generalized Appell polynomials (or their generalizations), in this case orthogonal bases in spaces of generalized functions are biorthogonal to the above-mentioned polynomials generalized functions. Over time the interest to the biorthogonal analysis went down because of the lack of interesting applications.

But methods developed within its framework, and some its results can be successfully used in another generalizations of the Gaussian analysis, in particular, in the Lévy analysis. For example, the proof of Theorem 2 is adopted from the biorthogonal analysis, cf. [24].

2.2 Interconnection between the Wick calculuses in the regular and nonregular cases

In the paper [15], in particular, a Wick product and Wick versions of holomorphic functions are introduced and studied on so-called *parametrized Kondratiev-type spaces of regular generalized functions* of the Lévy white noise analysis [15, 25]. As distinct from the Gaussian or Poissonian analysis, these spaces are not embedded into the spaces of nonregular generalized functions, but have with the last wide intersections (for example, (L^2) is a part of all these intersections). So, it is natural to consider a question about interconnection between the Wick calculuses on the spaces of regular and nonregular generalized functions. The answer is very simple: actually, on the above-mentioned intersections the Wick products and the Wick versions of holomorphic functions, introduced in [15] and in this paper, coincide. Now we'll explain this in detail.

Definition 5. Accept on default $\beta \in [0, 1]$. *Parametrized Kondratiev-type spaces of regular generalized functions* $(L^2)_{-q}^{-\beta}$ and $(L^2)^{-\beta}$ can be defined as follows: $(L^2)_{-q}^{-\beta}$ consists of formal series (5) such that $\|F\|_{(L^2)_{-q}^{-\beta}}^2 = \sum_{n=0}^{\infty} (n!)^{1-\beta} 2^{-qn} |F^{(n)}|_{ext}^2 < \infty$; $(L^2)^{-\beta} := \text{ind} \lim_{q \rightarrow \infty} (L^2)_{-q}^{-\beta}$.

The well-posedness of this definition is proved in [15, 25]. Note that the space of square integrable random variables $(L^2) = (L^2)_0^0$ is densely and continuously embedded into each $(L^2)_{-q}^{-\beta}$ and therefore into $(L^2)^{-\beta}$.

Remark 10. Let $(L^2)_{q, \tau}^{\beta}, (L^2)^{\beta} = \text{pr} \lim_{q \rightarrow \infty} (L^2)_{q, \tau}^{\beta}$ be parametrized Kondratiev-type spaces of regular test functions [15, 25], i.e., the positive spaces of a chain $(L^2)^{-\beta} \supset (L^2)_{-q}^{-\beta} \supset (L^2) \supset (L^2)_q^{\beta} \supset (L^2)^{\beta}$. It is not difficult to understand that $(L^2)_{q, \tau}^{\beta}$ consist of elements of form (5) such that $\|F\|_{(L^2)_{q, \tau}^{\beta}}^2 = \sum_{n=0}^{\infty} (n!)^{1+\beta} 2^{qn} |F^{(n)}|_{ext}^2 < \infty$. By analogy one can introduce spaces $(\mathcal{H}_{\tau})_q^{\beta}$ that consist of formal series (8) such that $\|f\|_{(\mathcal{H}_{\tau})_q^{\beta}}^2 = \sum_{n=0}^{\infty} (n!)^{1+\beta} 2^{qn} |f^{(n)}|_{\tau}^2 < \infty$. It is possible to study properties of these spaces and of its projective limits, to introduce and to study operators and operations on them; such considerations are interesting by itself and can be useful for applications. But, in contrast to the Gaussian and Poissonian analysis, in the Lévy analysis $(\mathcal{H}_{\tau})_q^{\beta} \not\subset (L^2)$ if $\beta < 1$, generally speaking, so, we cannot consider $(\mathcal{H}_{\tau})_q^{\beta}$ with $\beta < 1$ as spaces of test functions.

Definition 6 ([15]). For $F \in (L^2)^{-\beta}$ we define an \tilde{S} -transform $(\tilde{S}F)(\lambda)$, $\lambda \in \mathcal{D}_{\mathbb{C}}$, as a formal series

$$(\tilde{S}F)(\lambda) := \sum_{m=0}^{\infty} (F^{(m)}, \lambda^{\otimes m})_{ext} \equiv F^{(0)} + \sum_{m=1}^{\infty} (F^{(m)}, \lambda^{\otimes m})_{ext}, \quad (29)$$

where $F^{(m)} \in \mathcal{H}_{ext}^{(m)}$ are the kernels from decomposition (5) for F (cf. (15)). In particular, $(\tilde{S}F)(0) = F^{(0)}$, $\tilde{S}1 \equiv 1$.

Definition 7 ([15]). For $F, G \in (L^2)^{-\beta}$ and a holomorphic at $(\tilde{S}F)(0)$ function $h : \mathbb{C} \rightarrow \mathbb{C}$ we define a Wick product $F \tilde{\diamond} G$ and a Wick version $h^{\tilde{\diamond}}(F)$ by setting formally (cf. (16))

$$F \tilde{\diamond} G := \tilde{S}^{-1}(\tilde{S}F \cdot \tilde{S}G), \quad h^{\tilde{\diamond}}(F) := \tilde{S}^{-1}h(\tilde{S}F). \quad (30)$$

As in the nonregular case, the Wick multiplication $\tilde{\diamond}$ is commutative, associative and distributive over a field \mathbb{C} , and the following statement is fulfilled (cf. Theorem 2).

Theorem 3 ([15]). 1) Let $F_1, \dots, F_n \in (L^2)^{-\beta}$. Then $F_1 \tilde{\diamond} \dots \tilde{\diamond} F_n \in (L^2)^{-\beta}$. Moreover, the Wick multiplication is continuous in the sense that for any $q, q' \in \mathbb{Z}_+$ such that $F_1, \dots, F_n \in (L^2)^{-q'}$ and $q > q' + (1 - \beta) \log_2 n + 1$

$$\|F_1 \tilde{\diamond} \dots \tilde{\diamond} F_n\|_{(L^2)^{-q}} \leq \sqrt{\max_{m \in \mathbb{Z}_+} [2^{-m} (m+1)^{n-1}]} \|F_1\|_{(L^2)^{-q'}} \dots \|F_n\|_{(L^2)^{-q'}}$$

(cf. (26)). 2) Let $F \in (L^2)^{-\beta}$ and a function $h : \mathbb{C} \rightarrow \mathbb{C}$ be holomorphic at $(\tilde{S}F)(0)$. Then $h^{\tilde{\diamond}}(F) \in (L^2)^{-1}$.

Remark 11. Theorem 3 can be proved with the use of "coordinate formulas" for the Wick product and for the Wick versions of holomorphic functions on the spaces $(L^2)^{-\beta}$ [15]. Formally these formulas coincide with the corresponding formulas in the nonregular case, see Proposition 2. Actually, this coincidence is not accidental: the restriction of the multiplication \diamond to the spaces $\mathcal{H}_{ext}^{(n)}$, $n \in \mathbb{Z}_+$, is an analog of the symmetric tensor multiplication on these spaces, the proof of this fact coincides up to obvious modifications with the proof of the corresponding statement in the real case [31].

Comparing (15) with (29), (16) with (30), and taking into account Theorems 2 and 3, we obtain the following statement.

Theorem 4. 1) Let $F_1, \dots, F_n \in (\mathcal{H}_{-\tau}) \cap (L^2)^{-\beta}$. Then

$$F_1 \diamond \dots \diamond F_n = F_1 \tilde{\diamond} \dots \tilde{\diamond} F_n \in (\mathcal{H}_{-\tau}) \cap (L^2)^{-\beta}.$$

2) Let $F \in (\mathcal{H}_{-\tau}) \cap (L^2)^{-\beta}$ and a function $h : \mathbb{C} \rightarrow \mathbb{C}$ be holomorphic at $(SF)(0) = (\tilde{S}F)(0)$. Then $h^{\diamond}(F) = h^{\tilde{\diamond}}(F) \in (\mathcal{H}_{-\tau}) \cap (L^2)^{-1}$.

2.3 Interconnection between the Wick calculus and operators of stochastic differentiation

As is well known, a very important role in the Gaussian white noise analysis and its generalizations belongs to the extended stochastic integral and to its adjoint operator — the Hida stochastic derivative. Together with these operators, it is natural and useful to introduce and to study so-called *operators of stochastic differentiation*, which are closely related with the stochastic integral and derivative. Roughly speaking, one can understand the stochastic differentiation as a "differentiation" with respect to a "stochastic argument", i.e., the operator of stochastic differentiation acts on an orthogonal decomposition of a (generalized) random variable in common with an action of the differentiation operator on Taylor's decomposition of a function. The operators of stochastic differentiation can be used, in particular, in order to study some properties of the extended stochastic integral and of solutions of stochastic equations with Wick-type nonlinearities.

As is known [3], in the Gaussian analysis the operator of stochastic differentiation of order 1 is a differentiation (i.e., satisfies the Leibniz rule) with respect to the Wick multiplication. This important for applications property holds true in a Gamma-analysis (i.e., a white noise analysis connected with a so-called Gamma-measure) [22], in a Meixner analysis [26, 27], and in the Lévy analysis on the spaces of regular generalized functions [15]. But, in contrast to the

Gaussian case, in the Lévy analysis (in the same way as in the Gamma- and Meixner analysis) the operators of stochastic differentiation (in the same way as the Hida stochastic derivative) cannot be naturally continued from (L^2) to the spaces of nonregular generalized functions, see [30] for details. Nevertheless, one can introduce on these spaces natural analogs of the above-mentioned operators. These analogs are introduced and studied (in a real case) in [30]. They have properties similar to properties of "classical" operators of stochastic differentiation [13], and can be accepted as operators of stochastic differentiation on the spaces of nonregular generalized functions. Now we'll recall the definition of such operator of order 1, and will show that this operator satisfies the Leibniz rule with respect to the Wick multiplication \diamond .

Let $F_{ext}^{(m)} \in \mathcal{H}_{-\tau, \mathbb{C}}^{(m)}$, $m \in \mathbb{N} \setminus \{1\}$, $g \in \mathcal{H}_{\tau, \mathbb{C}}$. We define a generalized partial pairing $\langle F_{ext}^{(m)}, g \rangle_{ext} \in \mathcal{H}_{-\tau, \mathbb{C}}^{(m-1)}$ by setting for any $f^{(m-1)} \in \mathcal{H}_{\tau, \mathbb{C}}^{\widehat{\otimes} m-1}$

$$\langle \langle F_{ext}^{(m)}, g \rangle_{ext}, f^{(m-1)} \rangle_{ext} = \langle F_{ext}^{(m)}, g \widehat{\otimes} f^{(m-1)} \rangle_{ext}. \quad (31)$$

Since by the generalized Cauchy-Bunyakovsky inequality

$$|\langle F_{ext}^{(m)}, g \widehat{\otimes} f^{(m-1)} \rangle_{ext}| \leq |F_{ext}^{(m)}|_{\mathcal{H}_{-\tau, \mathbb{C}}^{(m)}} |g \widehat{\otimes} f^{(m-1)}|_{\tau} \leq |F_{ext}^{(m)}|_{\mathcal{H}_{-\tau, \mathbb{C}}^{(m)}} |g|_{\tau} |f^{(m-1)}|_{\tau},$$

this definition is well posed and

$$|\langle F_{ext}^{(m)}, g \rangle_{ext}|_{\mathcal{H}_{-\tau, \mathbb{C}}^{(m-1)}} \leq |F_{ext}^{(m)}|_{\mathcal{H}_{-\tau, \mathbb{C}}^{(m)}} |g|_{\tau}. \quad (32)$$

Definition 8. Let $g \in \mathcal{H}_{\tau, \mathbb{C}}$. We define (the analog of) the operator of stochastic differentiation

$$(D \circ)(g) : (\mathcal{H}_{-\tau}) \rightarrow (\mathcal{H}_{-\tau}) \quad (33)$$

as a linear continuous operator that is given by the formula

$$(DF)(g) := \sum_{m=1}^{\infty} m: \langle \circ^{\otimes m-1}, \langle F_{ext}^{(m)}, g \rangle_{ext} \rangle :, \quad (34)$$

where $F_{ext}^{(m)} \in \mathcal{H}_{-\tau, \mathbb{C}}^{(m)}$ are the kernels from decomposition (12) for $F \in (\mathcal{H}_{-\tau})$.

The proof of the well-posedness of this definition is based on estimate (32) and coincides up to obvious modifications with the proof of the corresponding statement in a real case [30].

Let us define a characterization set of the space $(\mathcal{H}_{-\tau})$ in terms of the S -transform, setting $B_{\tau} := S(\mathcal{H}_{-\tau}) \equiv \{SF : F \in (\mathcal{H}_{-\tau})\}$ (cf. [15]). It is clear that B_{τ} is a linear space, which consists of formal series $\sum_{m=0}^{\infty} \langle F_{ext}^{(m)}, \cdot^{\otimes m} \rangle_{ext}$ (see (15)) with the kernels $F_{ext}^{(m)} \in \mathcal{H}_{-\tau, \mathbb{C}}^{(m)}$ satisfying a condition: there exists $q \in \mathbb{N}_{q_0(\tau)} \subseteq \mathbb{Z}_+$ such that $\sum_{m=0}^{\infty} 2^{-qm} |F_{ext}^{(m)}|_{\mathcal{H}_{-\tau, \mathbb{C}}^{(m)}}^2 < \infty$. It follows from Definition 4 and Theorem 2 that B_{τ} is an algebra with respect to the pointwise multiplication. Moreover, if we introduce on B_{τ} a topology induced by the topology of $(\mathcal{H}_{-\tau})$, then the S -transform becomes a topological isomorphism between a topological algebra $(\mathcal{H}_{-\tau})$ with the Wick multiplication and a topological algebra B_{τ} with the pointwise multiplication.

Denote by

$$d_g : B_{\tau} \rightarrow B_{\tau}, \quad g \in \mathcal{H}_{\tau, \mathbb{C}}, \quad (35)$$

a directional derivative, i.e., for $(SF)(\cdot) = \sum_{m=0}^{\infty} \langle F_{ext}^{(m)}, \cdot^{\otimes m} \rangle_{ext} = \sum_{m=0}^{\infty} \langle U_m F_{ext}^{(m)}, \cdot^{\otimes m} \rangle \in B_\tau$ (see (15), (20); $F \in (\mathcal{H}_{-\tau})$, $F_{ext}^{(m)} \in \mathcal{H}_{-\tau, \mathbb{C}}^{(m)}$ are the kernels from decomposition (12) for F)

$$\begin{aligned} d_g(SF)(\cdot) &= \sum_{m=1}^{\infty} m \langle U_m F_{ext}^{(m)}, g \widehat{\otimes} (\cdot^{\otimes m-1}) \rangle = \sum_{m=1}^{\infty} m \langle F_{ext}^{(m)}, g \widehat{\otimes} (\cdot^{\otimes m-1}) \rangle_{ext} \\ &= \sum_{m=1}^{\infty} m \langle (F_{ext}^{(m)}, g)_{ext}, \cdot^{\otimes m-1} \rangle_{ext} = (S(DF)(g))(\cdot) \in B_\tau \end{aligned} \quad (36)$$

(see (20), (31), (34) and (15)). As we see, directional derivative (35) is the image on B_τ of operator of stochastic differentiation (33) under the S -transform (in particular, (35) is a linear *continuous* operator). Vice versa, operator of stochastic differentiation (33) is a pre-image of directional derivative (35) under the S -transform, i.e., for all $F \in (\mathcal{H}_{-\tau})$ and $g \in \mathcal{H}_{\tau, \mathbb{C}}$

$$(DF)(g) = S^{-1} d_g SF \in (\mathcal{H}_{-\tau}). \quad (37)$$

Now we are ready to prove the main result of this subsection.

Theorem 5. *Operator of stochastic differentiation (33) is a differentiation (i.e., satisfies the Leibniz rule) with respect to the Wick multiplication, i.e., for all $F, G \in (\mathcal{H}_{-\tau})$ and $g \in \mathcal{H}_{\tau, \mathbb{C}}$*

$$(D(F \diamond G))(g) = (DF)(g) \diamond G + F \diamond (DG)(g) \in (\mathcal{H}_{-\tau}). \quad (38)$$

Proof. First we note that the expressions in the left hand side and in the right hand side of (38) belong to $(\mathcal{H}_{-\tau})$, this follows from the definition of operator (33) and Theorem 2. As for equality (38), using (37), (16), the fact that the directional derivative satisfies the Leibniz rule, and (36), we obtain

$$\begin{aligned} (D(F \diamond G))(g) &= S^{-1} d_g (S(F \diamond G)) = S^{-1} d_g (SF \cdot SG) = S^{-1} [(d_g SF) \cdot SG + SF \cdot (d_g SG)] \\ &= S^{-1} [(S(DF)(g)) \cdot SG + SF \cdot (S(DG)(g))] = (DF)(g) \diamond G + F \diamond (DG)(g), \end{aligned}$$

which is what had to be proved. \square

Corollary. *Let $F \in (\mathcal{H}_{-\tau})$, $g \in \mathcal{H}_{\tau, \mathbb{C}}$, and $h : \mathbb{C} \rightarrow \mathbb{C}$ be a holomorphic at $(SF)(0)$ function. Then*

$$(Dh^\diamond(F))(g) = h^\diamond(F) \diamond (DF)(g) \in (\mathcal{H}_{-\tau}), \quad (39)$$

where h^\diamond is the Wick version of the usual derivative of a function h .

Proof. Using (38), one can prove by the mathematical induction method that for each $m \in \mathbb{Z}_+$

$$(D(F - (SF)(0))^{\diamond m})(g) = m(F - (SF)(0))^{\diamond m-1} \diamond (DF)(g). \quad (40)$$

Further, let $h_N^\diamond(F)$, $N \in \mathbb{N}$, be the Wick version of the N -th partial sum of decomposition (17) for h , i.e., $h_N^\diamond(F) = \sum_{m=0}^N h_m (F - (SF)(0))^{\diamond m}$, see (18). It follows from the linearity of the operator D , (40), Theorem 2 and Remark 8 that

$$\begin{aligned} (Dh_N^\diamond(F))(g) &= \sum_{m=0}^N h_m (D(F - (SF)(0))^{\diamond m})(g) \\ &= \sum_{m=1}^N h_m m (F - (SF)(0))^{\diamond m-1} \diamond (DF)(g) \xrightarrow{N \rightarrow \infty} h^\diamond(F) \diamond (DF)(g) \end{aligned}$$

in $(\mathcal{H}_{-\tau})$. On the other hand, it follows from Remark 8 and the continuity of the operator $(D \circ)(g)$ on $(\mathcal{H}_{-\tau})$ that $(Dh_N^\diamond(F))(g) \rightarrow (Dh^\diamond(F))(g)$ as $N \rightarrow \infty$ in $(\mathcal{H}_{-\tau})$. Therefore equality (39) is valid. \square

In a forthcoming paper we'll consider an interconnection between the Wick calculus and the stochastic integration on the spaces of nonregular generalized functions, and give examples of integral stochastic equations with Wick-type nonlinearities.

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Качановський М.О. Про Віківське числення на просторах нерегулярних узагальнених функцій аналізу білого шуму Леві // Карпатські матем. публ. — 2018. — Т.10, №1. — С. 114–132.

Розвиток теорії основних і узагальнених функцій, що залежать від нескінченної кількості змінних, є важливою та актуальною задачею, яка обумовлена потребами фізики і математики. Один з успішних підходів до побудови такої теорії полягає у введенні просторів вищезгаданих функцій таким чином, що дуальне спарювання між основними і узагальненими функціями породжується інтегруванням за деякою ймовірнісною мірою. Спочатку це була гауссівська міра, згодом були зроблені численні узагальнення. Зокрема, важливі результати можна отримати, використовуючи міру білого шуму Леві, відповідна теорія називається *аналізом білого шуму Леві*.

У гауссівському випадку можна будувати простори основних і узагальнених функцій та вводити деякі важливі оператори (наприклад, стохастичні інтеграли і похідні) на цих просторах за допомогою так званої *властивості хаотичного розкладу* (ВХР): грубо кажучи, кожен квадратично інтегровний випадковий величину можна розкласти у ряд повторних стохастичних інтегралів Іто від невідповідних функцій. У аналізі Леві нема ВХР, але є різні узагальнення цієї властивості.

У цій статті ми маємо справу з одним з найбільш корисних і перспективних узагальнень ВХР у аналізі Леві, запропонованим Є. В. Литвиновим, та з відповідними просторами нерегулярних узагальнених функцій. Метою статті є увести природний добуток (віківський добуток) на цих просторах, та вивчити деякі пов'язані питання. Основними результатами є теореми про властивості віківського добутку і віківських версій голоморфних функцій. Зокрема, ми доводимо, що оператор стохастичного диференціювання задовольняє правило Лейбніца відносно віківського множення. Крім того, ми показуємо, що віківські добутки і віківські версії голоморфних функцій, визначені на просторах регулярних і нерегулярних узагальнених функцій, побудованих за допомогою литвинівського узагальнення ВХР, співпадають на перетинах цих просторів.

Наші дослідження є внеском у подальший розвиток аналізу білого шуму Леві.

Ключові слова і фрази: процес Леві, віківський добуток, стохастичне диференціювання.



KURYLIAK A.O., TSVIGUN V.L.

WIMAN'S INEQUALITY FOR ANALYTIC FUNCTIONS IN $\mathbb{D} \times \mathbb{C}$ WITH RAPIDLY OSCILLATING COEFFICIENTS

Let \mathcal{A}^2 be a class of analytic functions f represented by power series of the form

$$f(z) = f(z_1, z_2) = \sum_{n+m=0}^{+\infty} a_{nm} z_1^n z_2^m$$

with the domain of convergence $\mathbb{T} = \{z \in \mathbb{C}^2: |z_1| < 1, |z_2| < +\infty\}$ such that $\frac{\partial}{\partial z_2} f(z_1, z_2) \not\equiv 0$ in \mathbb{T} and there exists $r_0 = (r_1^0, r_2^0) \in [0, 1) \times [0, +\infty)$ such that for all $r \in (r_1^0, 1) \times (r_2^0, +\infty)$ we have $r_1 \frac{\partial}{\partial r_1} \ln \mathfrak{M}_f(r) + \ln r_1 > 1$, where $\mathfrak{M}_f(r) = \sum_{n+m=0}^{+\infty} |a_{nm}| r_1^n r_2^m$. Let $\mathcal{K}(f, \theta) = \{f(z, t) = \sum_{n+m=0}^{+\infty} a_{nm} e^{2\pi i t(\theta_n + \theta_m)} : t \in \mathbb{R}\}$ be class of analytic functions, where (θ_{nm}) is a sequence of positive integer such that its arrangement (θ_k^*) by increasing satisfies the condition

$$\theta_{k+1}^* / \theta_k^* \geq q > 1, k > 0.$$

For analytic functions from the class $\mathcal{K}(f, \theta)$ Wiman's inequality is improved.

Key words and phrases: Wiman's type inequality, analytic functions of several variables.

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1 INTRODUCTION

In this paper we consider some analog of the classical inequality of A.Wiman (in this regard, see [1–7]) for the class \mathcal{A}_0^2 of analytic functions f represented by power series of the form

$$f(z) = f(z_1, z_2) = \sum_{n+m=0}^{+\infty} a_{nm} z_1^n z_2^m, \quad z = (z_1, z_2) \in \mathbb{C}^2, \quad (1)$$

with the domain of convergence $\mathbb{T} = \mathbb{D} \times \mathbb{C} = \{z \in \mathbb{C}^2: |z_1| < 1, |z_2| < +\infty\}$. Let \mathcal{A}_1^2 be the class of functions $f \in \mathcal{A}_0^2$ such that

$$\frac{\partial}{\partial z_2} f(z_1, z_2) \not\equiv 0 \quad (2)$$

in \mathbb{T} , \mathcal{A}_2^2 be the class of functions $f \in \mathcal{A}_0^2$ there exists $r_0 = (r_1^0, r_2^0) \in T := [0, 1) \times [0, +\infty)$ such that for all $r \in (r_1^0, 1) \times (r_2^0, +\infty)$ we have

$$r_1 \frac{\partial}{\partial r_1} \ln \mathfrak{M}_f(r) + \ln r_1 > 1, \quad \mathfrak{M}_f(r) := \sum_{n+m=0}^{+\infty} |a_{nm}| r_1^n r_2^m \quad (3)$$

and $\mathcal{A}^2 = \mathcal{A}_1^2 \cap \mathcal{A}_2^2$.

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Proposition 1. 1. If $f \in \mathcal{A}_0^2 \setminus \mathcal{A}_1^2$ then for every $\delta > 0$ there exists a set $E = E_f(\delta) := E_1 \times [1, r_2^0]$, $\int_{E_1 \cap [r_1^0, 1)} d \ln r_1 < +\infty$, such that for all $r \in T \setminus E$ the inequality

$$\mathfrak{M}_f(r) \leq \frac{\mu_f(r)}{(1-r)^{1+\delta}} \ln^{1/2+\delta} \frac{\mu_f(r)}{1-r}, \quad \mu_f(r) := \max\{|a_{nm}|r_1^n r_2^m : n, m \geq 0\}, \quad (4)$$

holds.

2. If $f \in \mathcal{A}_0^2 \setminus (\mathcal{A}_1^2 \cup \mathcal{A}_2^2)$ then for all $r \in T$, $\mathfrak{M}_f(r) \leq C < +\infty$.

Proof. 1. Remark that every function $f \in \mathcal{A}_0^2 \setminus \mathcal{A}_1^2$ is the function of the form $f(z) \equiv f_1(z_1)$ for all $z_2 \in \mathbb{C}$, i.e. is identical function of z_2 and analytic function of $z_1 \in \mathbb{D}$. Therefore the result of T.Kővari (see [8,9]) implies that inequality (4) holds for all $r \in T \setminus E$.

2. Further, $r_1 \frac{\partial}{\partial r_1} \ln \mathfrak{M}_f(r) = r_1 \frac{\partial}{\partial r_1} \ln \mathfrak{M}_{f_1}(r_1)$, where $\mathfrak{M}_{f_1}(r_1) = \sum_{n=0}^{+\infty} |a_{n0}|r_1^n$. Well known that the function $r_1 \frac{\partial}{\partial r_1} \ln \mathfrak{M}_{f_1}(r_1)$ is nondecreasing on $(0, 1)$. Therefore, with the denial of inequality (3) we obtain that the inequality $r_1 \frac{\partial}{\partial r_1} \ln \mathfrak{M}_{f_1}(r_1) + \ln r_1 \leq 1$ for all $r_1 \in (0, 1)$ holds. Hence, $\mathfrak{M}_{f_1}(r_1) = O(1)$ ($r_1 \rightarrow 1 - 0$). \square

Remark 1. For the function $f \in \mathcal{A}_0^2 \setminus \mathcal{A}_2^2$ similarly as in proof of 2) we obtain

$$\ln \mathfrak{M}_f(r_1, r_2) - \ln \mathfrak{M}_f(r_1^0, r_2) \leq \frac{1}{2}(2 - \ln r_1 r_1^0) \ln \frac{r_1}{r_1^0} < (1 - \ln r_1^0) \ln \frac{1}{r_1^0},$$

for all $(r_1, r_2) \in (r_1^0, 1) \times (0, +\infty)$.

For $r = (r_1, r_2) \in T$ and a function $f \in \mathcal{A}^2$ we denote

$$\begin{aligned} \Delta_r &= \{(t_1, t_2) \in T : t_1 > r_1, t_2 > r_2\}, \\ M_f(r) &= \max\{|f(z)| : |z_1| \leq r_1, |z_2| \leq r_2\}, \\ \mu_f(r) &= \max\{|a_{nm}|r_1^n r_2^m : (n, m) \in \mathbb{Z}_+^2\}. \end{aligned}$$

We call $E \subset T$ a set of *asymptotically finite logarithmic measure* on T ($E \in \mathcal{Y}$) if there exists $R \in T$ such that

$$\nu_{\ln}(E \cap \Delta_R) := \iint_{E \cap \Delta_R} \frac{dr_1 dr_2}{(1-r_1)r_2} < +\infty,$$

i.e. the set $E \cap \Delta_R$ is a set of *finite logarithmic measure* on T .

We note that for a function $f \in \mathcal{A}^2$ of the form $f(z) = f_1(z_1) \cdot f_2(z_2)$, where f_1 is analytical in \mathbb{D} and f_2 is entire function of one variable, the inequality

$$M_f(r) \leq \mathfrak{M}_f(r) \leq \frac{\mu_f(r)}{(1-r_1)^{1+\delta}} \ln^{1+\delta} \frac{\mu_f(r)}{1-r_1} \ln^{1/2+\delta} r_2 \quad (5)$$

for every $r \in \Delta_{r^0} \setminus E$, $E = E_1 \times E_2 \subset T$,

$$\int_{E_1 \cap (0,1)} \frac{dr_1}{1-r_1} < +\infty, \quad \int_{E_2 \cap (1,+\infty)} \frac{dr_2}{r_2} < +\infty,$$

follows from classical Wiman's inequality [6]

$$\mathfrak{M}_{f_2}(r_2) \leq \mu_{f_2}(r_2) (\ln \mu_{f_2}(r_2))^{1/2} \ln^{1/2} r_2 \left(r_2 \in (r_2^0, +\infty) \setminus E_2, \int_{E_2 \cap (1,+\infty)} \frac{(\ln r)^{1/2} dr}{r} < +\infty \right)$$

for entire function f_2 and Kővari inequality [8]

$$\mathfrak{M}_{f_1}(r_1) \leq \frac{\mu_{f_1}(r_1)}{(1-r_1)^{1+\delta}} \ln^{1/2+\delta} \frac{\mu_{f_1}(r_1)}{1-r_1} \left(r_1 \in (r_1^0, 1) \setminus E_1, \int_{E_1 \cap (0,1)} \frac{dr}{1-r} < +\infty \right)$$

for analytic in \mathbb{D} function f_1 , where $\mathfrak{M}_g(t) = \sum_{n=0}^{+\infty} |g_n|t^n$, $\mu_g(t) = \max\{|g_n|t^n : n \geq 0\}$ and function $g(\tau) = \sum_{n=0}^{+\infty} g_n \tau^n$ and $t > 0$. Moreover, $v_{\ln}(E \cap \Delta_R) < +\infty$ for every $R = (R_1, R_2) \in T, R_1 > 0, R_2 > 0$.

Inequality (5) for the class \mathcal{A}^2 is proved in [10].

Theorem 1 ([10]). *Let $f \in \mathcal{A}^2$. For every $\delta > 0$ there exists a set $E = E(\delta, f) \in \mathcal{Y}$ such that for $r \in T \setminus E$ inequality (5) holds.*

None of the exponents $1 + \delta$ of (5) can not be replaced by a number less than 1 (see [10]).

Remark 2. Remark, that inequality (5) follows from Proposition 1 also in the cases $f \in \mathcal{A}_0^2 \setminus (\mathcal{A}_1^2 \cup \mathcal{A}_2^2)$, $f \in \mathcal{A}_0^2 \setminus \mathcal{A}_1^2$, i.e., analog of Wiman's inequality is not considered only in the case $f \in \mathcal{A}_1^2 \setminus \mathcal{A}_2^2$.

Let $\Omega = [0, 1]$ and P be the Lebesgue measure on \mathbb{R} . We consider the Steinhaus probability space (Ω, \mathcal{A}, P) , where \mathcal{A} is the σ -algebra of Lebesgue measurable subsets of Ω .

Let $Z = (Z_{nm}(t))$ be some sequence of complex valued random variables defined in this space. For $f \in \mathcal{A}^2$ by $\mathcal{K}(f, Z)$ we denote the class of random analytic functions of the form

$$f(z, t) = \sum_{n+m=0}^{+\infty} a_{nm} Z_{nm}(t) z_1^n z_2^m. \tag{6}$$

In the sequel, the notion "almost surely" will be used in the sense that the corresponding property holds *almost everywhere* with respect to Lebesgue measure P on Ω . We say that some relation holds *almost surely in the class $\mathcal{K}(f, Z)$* if it holds for each analytic function $f(z, t)$ of the form (6) almost surely in t .

Let $Z = (Z_{nm}(t))$ be some sequence of random variables defined in this space. $Z_{nm}(t) = X_{nm}(t) + iY_{nm}(t)$ such that both $X = X_{nm}(t)$ and $Y = Y_{nm}(t)$ are real multiplicative system (MS). For $f \in \mathcal{A}^2$ by $\mathcal{K}(f, Z)$ we denote the class of random analytic functions of the form

$$f(z, t) = \sum_{n+m=0}^{+\infty} a_{nm} Z_{nm}(t) z_1^n z_2^m. \tag{7}$$

For such functions in [11] it is proved following statement (Levy's phenomenon).

Theorem ([11]). *If $f \in \mathcal{A}^2$, and $Z = (Z_{nm}(t))$, $Z_{nm}(t) = X_{nm}(t) + iY_{nm}(t)$ such that $X = (X_{nm}(t))$ and $Y = (Y_{nm}(t))$ are real multiplicative systems uniformly bounded by the number 1, then for every $\delta > 0$ almost surely in $\mathcal{K}(f, Z)$ there exists a set $E = E(f, t, \delta), E \in \mathcal{Y}$, such that for all $r \in T \setminus E$*

$$M_f(r, t) := \max\{|f(z, t)| : |z_1| \leq r_1, |z_2| \leq r_2\} \leq \frac{\mu_f(r)}{(1-r_1)^{1/2+\delta}} \ln^{1/2+\delta} \frac{\mu_f(r)}{1-r_1} \cdot \ln^{1/4+\delta} r_2. \tag{8}$$

In the case when $\mathcal{R} = (R_n(t))$ is the Rademacher sequence, i.e. $(R_n(t))$ is a sequence of independent uniformly distributed random variables on $[0, 1]$ such that $P\{t: R_n(t) = \pm 1\} = 1/2$, P. Levy [12] proved that for any entire function f of one complex variable we can replace the exponent $1/2$ by $1/4$ in the classical Wiman's inequality almost surely in the class $\mathcal{K}(f, \mathcal{R})$ (*Levy's phenomenon*). Later P. Erdős and A. Rényi [13] proved the same result for the class $\mathcal{K}(f, H)$, where $H = (e^{2\pi i \omega_n(t)})$ is the Steinhaus sequence, i.e. $(\omega_n(t))$ is a sequence of independent uniformly distributed random variables on $[0, 1]$. This statement is true also for any class $\mathcal{K}(f, X)$, where $X = (X_n(t))$ is multiplicative system (MS) uniformly bounded by the number 1. That is for all $n \in \mathbb{N}$ and $t \in [0, 1]$ we have $|X_n(t)| \leq 1$ and

$$\text{for all } 1 \leq i_1 < i_2 < \dots < i_k: \mathbf{M}(X_{i_1} X_{i_2} \dots X_{i_k}) = 0,$$

where $\mathbf{M}\xi$ is the expectation of a random variable ξ ([14, 15]). The same holds for $Z = (Z_n)$, $Z_n = X_n + iY_n$, and $X = (X_n), Y = (Y_n)$ are both MS.

In the spring of 1996 during the report of P. V. Filevych at the Lviv seminar of the theory of analytic functions professors A. A. Goldberg and M. M. Sheremeta posed the following question (see [16]). Does Levy's phenomenon take place for analogues of Wiman's inequality for entire functions of several complex variables?

In the papers [16, 17] we have found an affirmative answer to this question about Fenton's inequality [18] for random entire functions of two complex variables, in [19] about a inequality from [21] for random entire functions of several complex variables, in [27] in the case of analytic functions in the polydisc.

In this paper we consider the class $\mathcal{K}(f, \theta)$ of analytic functions

$$f(z, t) = f(z_1, z_2, t) = \sum_{n+m=0}^{+\infty} a_{nm} e^{2\pi i \theta_{nm} t} z_1^n z_2^m. \tag{9}$$

Here (θ_{nm}) is a sequence of positive integer such that its arrangement (θ_k^*) by increasing $\{(n, m) \in \mathbb{Z}_+^2\} = \{\theta_k^* : k \in \mathbb{Z}_+\}$, $\theta_{k+1}^* > \theta_k^*$, satisfies the condition (θ is Hadamard sequence)

$$\theta_{k+1}^* / \theta_k^* \geq q > 1, k > 0. \tag{10}$$

Remark, that in the case $q \geq 2$ analytic functions of the form (9) satisfy the assumptions of previous theorem from [11], because $(\cos \theta_n t), (\sin \theta_n t)$ are MS. But in the case $q > 1$ the sequence of random variables $(\cos \theta_n t)_{n \in \mathbb{Z}_+}$ need not be a MS (see [16]). So the following question arrives naturally: *does Levy's phenomenon hold for the class $\mathcal{K}(f, \theta)$ with $f \in \mathcal{A}^2$ and a Hadamard sequence θ ?*

2 MAIN RESULT

Theorem 2. *Let $\delta > 0, f \in \mathcal{K}(f, \theta)$ be an analytic function of the from (9) and a sequence of a positive integer $(\theta_{nm})_{(n,m) \in \mathbb{Z}_+^2}$ satisfies condition (10). Then almost surely for $t \in \mathbb{R}$ there exists $E(\delta, t) \in Y$ such that for all $r \in T \setminus E$ we have*

$$M_f(r, t) = \max_{|z|=r} |f(z, t)| \leq \frac{\mu_f(r)}{(1-r_1)^{1/2+\delta}} \ln^{1/2+\delta} \frac{\mu_f(r)}{1-r_1} \ln^{1/4+\delta} r_2. \tag{11}$$

Similar inequalities for entire functions of one complex variable one can find in [13, 26], for analytic functions in the unit disc in [9], for entire functions of two variables [11, 17, 19, 20, 22, 23, 27], for analytic functions without exceptional sets [15, 24].

3 AUXILIARY LEMMAS

Lemma 1 ([25]). *Let $\theta = (\theta_{nm})$ be a sequence of integers which satisfies (10). Then for any $\beta > 0, l \in \mathbb{N}, l \geq 2$ and $\{c_{n,m} : (nm) \in \mathbb{Z}_+^2\} \subset \mathbb{C}$ there exists $A > 0, B > 0$ such that*

$$P \left\{ t : \max \left\{ \left| \sum_{n+m=0}^l c_{nm} e^{in\psi_1} e^{im\psi_2} e^{2\pi i \theta_{nm} t} \right| : \psi \in [0, 2\pi]^2 \right\} \geq A_\beta S_l \ln^{1/2} l \right\} \leq \frac{B}{l^\beta}, \quad (12)$$

where $S_l^2 = \sum_{n+m=0}^l |c_{nm}|^2$.

Lemma 2 ([10]). *Let $\delta > 0$ and $h : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ be an increasing function on each variable such that*

$$\int_1^{+\infty} \int_1^{+\infty} \frac{du_1 du_2}{h(u_1, u_2)} < +\infty.$$

Then there exists a set $E \subset T$ of asymptotically finite logarithmic measure such that for all $r \in T \setminus E$ we have

$$\frac{\partial}{\partial r_1} \ln \mathfrak{M}_f(r) \leq \frac{1}{1-r_1} h(\ln \mathfrak{M}_f(r), \ln r_2), \quad \frac{\partial}{\partial r_2} \ln \mathfrak{M}_f(r) \leq \frac{1}{r_2(1-r_1)^\delta} (\ln \mathfrak{M}_f(r))^{1+\delta}. \quad (13)$$

Lemma 3. *There exists set $E \in Y$ such that for all $r \in T \setminus E$ we have*

$$\sum_{n+m=0}^{+\infty} (n+m) |a_{nm}| r_1^n r_2^m \leq \frac{\mu_f(r)}{(1-r_1)^{2+\delta}} \ln^{2+3\delta} \frac{\mu_f(r)}{(1-r_1)} \ln^{3/2+3\delta} r_2.$$

Proof. Let $h(r) = (r_1 r_2)^{1+\delta}$. Then by Lemma 2, there exist set $E \in Y$ such that for all $r \in T \setminus E_1$ we have

$$\begin{aligned} \frac{\partial}{\partial r_1} \ln \mathfrak{M}_f(r) &= \frac{1}{\mathfrak{M}_f(r)} \sum_{n+m=0}^{+\infty} n |a_{nm}| r_1^{n-1} r_2^m = \frac{1}{r_1 \mathfrak{M}_f(r)} \sum_{n+m=0}^{+\infty} n |a_{nm}| r_1^n r_2^m \\ &\leq \frac{1}{1-r_1} \ln^{1+\delta} \mathfrak{M}_f(r) \ln r_2^{1+\delta} \sum_{n+m=0}^{+\infty} n |a_{nm}| r_1^n r_2^m \leq \frac{r_1}{1-r_1} \mathfrak{M}_f(r) \ln^{1+\delta} \mathfrak{M}_f(r) \ln r_2^{1+\delta} \\ &\leq \frac{1}{1-r_1} \mathfrak{M}_f(r) \ln^{1+\delta} \mathfrak{M}_f(r) \ln r_2^{1+\delta} \sum_{n+m=0}^{+\infty} m |a_{nm}| r_1^n r_2^m \leq \frac{r_2}{r_2(1-r_1)^\delta} \mathfrak{M}_f(r) \ln^{1+\delta} \mathfrak{M}_f(r) \\ &\leq \frac{1}{(1-r_1)^\delta} \mathfrak{M}_f(r) \ln^{1+\delta} \mathfrak{M}_f(r) \sum_{n+m=0}^{+\infty} (n+m) |a_{nm}| r_1^n r_2^m \leq \frac{2\mathfrak{M}_f(r)}{1-r_1} \ln^{1+\delta} \mathfrak{M}_f(r) \ln^{1+\delta} r_2. \end{aligned}$$

By Theorem 1 we obtain for all $r \in T \setminus E_2$

$$\begin{aligned} \sum_{n+m=0}^{+\infty} (n+m) |a_{nm}| r_1^n r_2^m &\leq \frac{\mu_f(r)}{(1-r_1)^{2+\delta}} \ln^{1+\delta} \frac{\mu_f(r)}{1-r_1} \ln^{1/2+\delta} r_2 \ln^{1+2\delta} \frac{\mu_f(r)}{1-r_1} \ln^{1+2\delta} r_2 \\ &\leq \frac{\mu_f(r)}{(1-r_1)^{2+\delta}} \ln^{2+3\delta} \frac{\mu_f(r)}{1-r_1} \ln^{3/2+3\delta} r_2. \end{aligned}$$

□

Proof of Theorem 2. For $k, m \in \mathbb{Z}_+$ and $l \in \mathbb{Z}$ such that $k > -l$ we denote

$$G_{kl} = \left\{ r = (r_1, r_2) \in T : k \leq \ln \frac{1}{1-r_1} \leq k+1, l \leq \ln \mu_f(r) \leq l+1 \right\},$$

$$G_{klm} = \left\{ r = (r_1, r_2) \in G_{kl} : m \leq \ln r_2 \leq m+1 \right\}, \quad G_{kl}^+ = \bigcup_{i=k}^{+\infty} \bigcup_{j=l}^{+\infty} G_{ij}.$$

Remark that the set

$$E_0 = \left\{ r \in T : \ln \frac{1}{1-r_1} + \ln \mu_f(r) < 1 \right\} = \left\{ r \in T : \frac{\mu_f(r)}{1-r_1} < e \right\} \in Y,$$

because there exists r_0 such that $E_0 \cap \Delta_{r_0} = \emptyset$. By Lemma 3 for all $r \in T \setminus E_1$ we have

$$\begin{aligned} \sum_{n+m \geq d} |a_{nm}| r_1^n r_2^m &\leq \sum_{n+m \geq d} \frac{n+m}{d} |a_{nm}| r_1^n r_2^m \leq \frac{1}{d} \sum_{n+m=0}^{+\infty} (n+m) |a_{nm}| r_1^n r_2^m \\ &\leq \frac{1}{d} \frac{\mu_f(r)}{(1-r_1)^{2+\delta}} \cdot \ln^{2+3\delta} \frac{\mu_f(r)}{1-r_1} \ln^{3/2+3\delta} r_2 \leq \mu_f(r), \end{aligned} \quad (14)$$

where $d = d(r) = \frac{e^{2+\delta}}{(1-r_1)^{2+\delta}} \cdot \ln^{2+3\delta} \frac{\mu_f(r)}{1-r_1} \ln^{3/2+3\delta} r_2$.

Let $G_{kl}^* = G_{kl} \setminus E_2$, $I = \{(i, j) : G_{ij}^* \neq \emptyset\}$, $E_2 = E_0 \cup E_1 \cup \left(\bigcup_{(i,j) \notin I} G_{ij} \right)$. Then $\#I = +\infty$. For $(k, l) \in I$ we choose a sequence $r^{(k,l)} \in G_{kl}^*$ such that $M_f(r^{(k,l)}) = \min_{r \in G_{kl}^*} M_f(r)$. So, for all $r \in G_{kl}^*$ we get

$$\begin{aligned} \frac{1}{e} \mu_f(r^{(k,l)}) &\leq \mu_f(r) \leq e \mu_f(r^{(k,l)}), \quad \frac{1}{e} \frac{1}{1-r_1^{(k,l)}} \leq \frac{1}{1-r_1} \leq e \frac{1}{1-r_1^{(k,l)}}, \\ \frac{1}{e^2} \frac{\mu_f(r^{(k,l)})}{1-r_1^{(k,l)}} &\leq \frac{\mu_f(r)}{1-r_1} \leq \frac{e^2 \mu_f(r^{(k,l)})}{1-r_1^{(k,l)}} \end{aligned} \quad (15)$$

and also $\bigcup_{(k,l) \in I} G_{kl}^* = \bigcup_{(k,l) \in I} G_{kl} \setminus E_2 = \bigcup_{k,l=1}^{+\infty} G_{kl} \setminus E_2 = T \setminus E_2$. Denote $N_{kl} = [2d_1(r^{(k,l)})]$, where

$$d_1(r) = \frac{e^{2+\delta}}{(1-r_1)^{2+\delta}} \cdot \ln^{2+3\delta} \frac{e^2 \mu_f(r)}{1-r_1} \ln^{3/2+3\delta} (er_2).$$

For $r \in G_{kl}^*$ we put

$$W_{N_{kl}}(r, t) = \max \left\{ \left| \sum_{n+m \leq N_{kl}} a_{nm} r_1^n r_2^m e^{in_1 \psi_1 + in_2 \psi_2 + 2\pi i \theta_{nm} t} \right| : \psi \in [0, 2\pi]^2 \right\}.$$

For a Lebesgue measurable set $G \subset G_{kl}^*$ and for $(k, l) \in I$ we denote $\nu_{kl}(G) = \frac{\text{meas}(G)}{\text{meas}(G_{kl}^*)}$, where meas denotes the Lebesgue measure on \mathbb{R}^2 .

Remark that ν_{kl} is a probability measure defined on the family of Lebesgue measurable subsets of G_{kl}^* ([19]). Let $\Omega = \bigcup_{(k,l) \in I} G_{kl}^*$ and for all $i, j \in \mathbb{Z}_+$ $k_i, l_{i,j} : (k_i, l_{i,j}) \in I$, $k_i < k_{i+1}$,

$l_{i,j} < l_{i,j+1}$. For Lebesgue measurable subsets G of Ω we denote

$$\nu(G) = 2^{k_0} \sum_{i=0}^{+\infty} \left(\frac{1}{2^{k_i}} \left(1 - \left(\frac{1}{2} \right)^{k_{i+1}-k_i} \right) \times \sum_{j=0}^{N_i} \frac{2^{l_{i,0}}}{2^{l_{i,j}}} \frac{\left(1 - \left(\frac{1}{2} \right)^{l_{i,j+1}-l_{i,j}} \right)}{1 - \left(\frac{1}{2} \right)^{l_{i,N_{i+1}}+l_{i,0}}} \nu_{k_{i+1}l_{i+1,j+1}}(G \cap G_{k_{j+1}l_{i+1,j+1}}^*) \right),$$

where $N_i = \max\{j : (k_i, l_{ij}) \in I\}$. Remark that $\nu_{k_{j+1}l_{j+1}}(G_{k_{j+1}l_{j+1}}^*) = \nu(\Omega) = 1$.

Thus ν is a probability measure, which is defined on measurable subsets of Ω . On $[0, 1] \times \Omega$ we define the probability measure $P_0 = P \otimes \nu$, which is a direct product of the probability measures P and ν . Now for $(k; l) \in I$ we define

$$F_{kl} = \{(t, r) \in [0, 1] \times \Omega : W_{N_{kl}}(r, t) > AS_{N_{kl}}(r) \ln^{1/2} N_{kl}\},$$

$$F_{kl}(r) = \{t \in [0, 1] : W_{N_{kl}}(r, t) > AS_{N_{kl}}(r) \ln^{1/2} N_{kl}\},$$

where $S_{N_{kl}}^2(r) = \sum_{n+m=0}^{N_{kl}} |a_{nm}|^2 r_1^{2n} r_2^{2m}$ and A is the constant from Lemma 1 with $\beta = 1$. Using Fubini's theorem and Lemma 1 with $c_{nm} = a_{nm} r_1^n r_2^m$ and $\beta = 1$, we get for $(k, l) \in I$

$$P_0(F_{kl}) = \int_{\Omega} \left(\int_{F_{kl}(r)} dP \right) d\nu = \int_{\Omega} P(F_{kl}(r)) d\nu \leq \frac{1}{N_{kl}} \nu(\Omega) = \frac{1}{N_{kl}}.$$

Note that $N_{kl} > \frac{1}{(1-r_1^{(k,l)})^{2+\delta}} \ln^{2+3\delta} \frac{\mu_f(r^{(k,l)})}{1-r_1^{(k,l)}} \ln^{3/2+3\delta} r_2^{(k,l)} \geq e^{2k} (l+k)^3$. Therefore

$$\sum_{(k,l) \in I} P_0(F_{kl}) \leq \sum_{k=1}^{+\infty} \sum_{l=-k+1}^{+\infty} \frac{1}{e^{2k} (l+k)^3} < +\infty.$$

By Borel-Cantelli's lemma the infinite quantity of the events $\{F_{kl} : (k, l) \in I\}$ may occur with probability zero. So,

$$P_0(F) = 1, \quad F = \bigcup_{s=1}^{+\infty} \bigcup_{m=1}^{+\infty} \bigcap_{\substack{k \geq s, l \geq m \\ (k,l) \in I}} \overline{F_{kl}} \subset [0, 1] \times \Omega.$$

Then for any point $(t, r) \in F$ there exist $k_0 = k_0(t, r)$ and $l_0 = l_0(t, r)$ such that for all $k \geq k_0$, $l \geq l_0$, $(k, l) \in I$ we have $W_{N_{kl}}(r, t) \leq AS_{N_{kl}}(r) \ln^{1/2} N_{kl}$.

So, $\nu(F^\wedge(t)) = 1$ (see [19]).

For any $t \in F_1$ ([19]) and $(k, l) \in I$ we choose a point $r_0^{(k,l)}(t) \in G_{kl}^*$ such that

$$W_{N_{kl}}(r_0^{(k,l)}(t), t) \geq \frac{3}{4} M_{kl}(t), \quad M_{kl}(t) \stackrel{\text{def}}{=} \sup\{W_{N_{kl}}(r, t) : r \in G_{kl}^*\}.$$

Then from $\nu_{kl}(F^\wedge(t) \cap G_{kl}^*) = 1$ for all $(k, l) \in I$ it follows that there exists a point $r^{(k,l)}(t) \in G_{kl}^* \cap F^\wedge(t)$ such that $|W_{N_{kl}}(r_0^{(k,l)}(t), t) - W_{N_{kl}}(r^{(k,l)}(t), t)| < \frac{1}{4} M_{kl}(t)$ or

$$\frac{3}{4} M_{kl}(t) \leq W_{N_{kl}}(r_0^{(k,l)}(t), t) \leq W_{N_{kl}}(r^{(k,l)}(t), t) + \frac{1}{4} M_{kl}(t).$$

Since $(t, r^{(k,l)}(t)) \in F$, from inequality (3) we obtain

$$\frac{1}{2}M_{kl}(t) \leq W_{N_{kl}}(r^{(k,l)}(t), t) \leq AS_{N_{kl}}(r^{(k,l)}(t)) \ln^{1/2} N_{kl}.$$

Now for $r^{(k,l)} = r^{(k,l)}(t)$ we get

$$S_{N_{kl}}^2(r^{(k,l)}) \leq \mu_f(r^{(k,l)}) \mathfrak{M}_f(r^{(k,l)}) \leq \frac{\mu_f^2(r^{(k,l)})}{(1-r_1^{(k,l)})^{1+\delta}} \ln^{1+\delta} \frac{\mu_f(r^{(k,l)})}{1-r_1^{(k,l)}} \ln^{1/2+\delta} r_2^{(k,l)}.$$

So, for $t \in F_1$ and all $k \geq k_0(t), l \geq l_0(t)$, we obtain

$$S_N(r^{(k,l)}) \leq \mu_f(r^{(k,l)}) \left(\frac{1}{1-r_1^{(k,l)}} \ln \frac{\mu_f(r^{(k,l)})}{1-r_1^{(k,l)}} \sqrt{\ln r_2^{(k,l)}} \right)^{1/2+\delta/2}. \quad (16)$$

It follows from (15) that $d_1(r^{(k,l)}) \geq d(r)$ for $r \in G_{kl}^*$. Then for $t \in F_1, r \in F^\wedge(t) \cap G_{kl}^*, (k,l) \in I, k \geq k_0(t), l \geq l_0(t)$ we get

$$M_f(r, t) \leq \sum_{n+m \geq 2d_1(r^{(k,l)})} |a_{nm}| r_1^n r_2^m + W_{N_{kl}}(r, t) \leq \sum_{n+m \geq 2d(r)} |a_{nm}| r_1^n r_2^m + M_{kl}(t).$$

Finally for $t \in F_1, r \in F^\wedge(t) \cap G_{kl}^*, l \geq l_0(t)$ and $k \geq k_0(t)$ we obtain

$$\begin{aligned} M_f(r^{(k,l)}, t) &\leq \mu_f(r^{(k,l)}) + 2AS_{N_{kl}}(r^{(k,l)}) \ln^{1/2} N_{kl} \\ &\leq \mu_f(r^{(k,l)}) + 2A\mu_f(r^{(k,l)}) \left(\frac{1}{1-r_1^{(k,l)}} \ln \frac{\mu_f(r^{(k,l)})}{1-r_1^{(k,l)}} \right)^{1/2+\delta/2} \ln^{1/4+\delta} r_2^{(k,l)} \\ &\quad \times \ln \left(\frac{2e^{2+\delta}}{(1-r_1^{(k,l)})^{2+\delta}} \cdot \ln^{2+3\delta} \frac{e^2 \mu_f(r^{(k,l)})}{1-r_1^{(k,l)}} \ln^{3/2+3\delta} (er_2^{(k,l)}) \right), \\ M_f(r, t) &\leq \frac{\mu_f(r)}{(1-r_1)^{1/2+\delta}} \cdot \ln^{1/2+\delta} \frac{\mu_f(r)}{1-r_1} \ln^{1/4+\delta} r_2. \end{aligned} \quad (17)$$

Therefore inequality (17) holds almost surely ($t \in F_1, P(F_1) = 1$) for all

$$r \in \left(\bigcup_{(k,l) \in I} (G_{kl}^* \cap F^\wedge(t)) \cap G_{kl}^+ \right) \setminus E^* = (T \cap G_{kl}^+) \setminus (E^* \cup G^* \cup E_1) = T \setminus E_2,$$

where $G_{kl}^+ = \bigcup_{i=k}^{+\infty} \bigcup_{j=l}^{+\infty} G_{kl}$, $E_2 = E_1 \cup G^* \cup E^*$, $G^* = \bigcup_{(k,l) \in I} (G_{kl}^* \setminus F^\wedge(t))$.

It remains to remark that $\nu(G^*)$ satisfies $\nu(G^*) = \sum_{(k,l) \in I} (\nu_{kl}(G_{kl}^*) - \nu_{kl}(F^\wedge(t))) = 0$. Then for all $(k,l) \in I$ we obtain

$$\nu_{kl}(G_{kl}^* \setminus F^\wedge(t)) = \frac{\text{meas}(G_{kl}^* \setminus F^\wedge(t))}{\text{meas}(G_{kl}^*)} = 0, \quad \text{meas}(G_{kl}^* \setminus F^\wedge(t)) = \iint_{G_{kl}^* \setminus F^\wedge(t)} \frac{dr_1 dr_2}{(1-r_1)r_2} = 0.$$

□

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Нехай \mathcal{A}^2 клас аналітичних функцій f вигляду

$$f(z) = f(z_1, z_2) = \sum_{n+m=0}^{+\infty} a_{nm} z_1^n z_2^m$$

з областю збіжності $\mathbb{T} = \{z \in \mathbb{C}^2: |z_1| < 1, |z_2| < +\infty\}$ таких, що $\frac{\partial}{\partial z_2} f(z_1, z_2) \not\equiv 0$ в \mathbb{T} і існує $r_0 = (r_1^0, r_2^0) \in [0, 1) \times [0, +\infty)$ таке, що для всіх $r \in (r_1^0, 1) \times (r_2^0, +\infty)$ маємо $r_1 \frac{\partial}{\partial r_1} \ln \mathfrak{M}_f(r) + \ln r_1 > 1$, де $\mathfrak{M}_f(r) = \sum_{n+m=0}^{+\infty} |a_{nm}| r_1^n r_2^m$. Нехай $\mathcal{K}(f, \theta) = \{f(z, t) = \sum_{n+m=0}^{+\infty} a_{nm} e^{2\pi i t(\theta_n + \theta_m)} : t \in \mathbb{R}\}$ — клас аналітичних функцій, де (θ_{nm}) — послідовність додатних цілих чисел така, що її впорядкування (θ_k^*) за зростанням задовольняє умову

$$\theta_{k+1}^* / \theta_k^* \geq q > 1, k > 0.$$

Для аналітичних функцій з класу $\mathcal{K}(f, \theta)$ уточнено нерівність типу Вімана.

Ключові слова і фрази: нерівність типу Вімана, аналітичні функції від декількох комплексних змінних.



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TRANSLATION, MODULATION AND DILATION SYSTEMS IN SET-VALUED SIGNAL PROCESSING

In this paper, we investigate a very important function space consists of set-valued functions defined on the set of real numbers with values on the space of all compact-convex subsets of complex numbers for which the p th power of their norm is integrable. In general, this space is denoted by $L^p(\mathbb{R}, \Omega(\mathbb{C}))$ for $1 \leq p < \infty$ and it has an algebraic structure named as a quasilinear space which is a generalization of a classical linear space. Further, we introduce an inner-product (set-valued inner product) on $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ and we think it is especially important to manage interval-valued data and interval-based signal processing. This also can be used in imprecise expectations. The definition of inner-product on $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ is based on Aumann integral which is ready for use integration of set-valued functions and we show that the space $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ is a Hilbert quasilinear space. Finally, we give translation, modulation and dilation operators which are three foundational set-valued operators on Hilbert quasilinear space $L^2(\mathbb{R}, \Omega(\mathbb{C}))$.

Key words and phrases: Hilbert quasilinear space, set-valued function, Aumann integral, translation, modulation, dilation.

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INTRODUCTION

The translation, modulation and dilation operators play an important role in signal processing. These operators are usually applied to electromagnetic signals such as radio, lasers, optics and computer networks. For example, the translation operator provides parallel displacement for a discrete-time signal. The modulation operator changes the wealths of a sound wave. As it is well known, converting an analog signal to a digital signal leads to ambiguous computation errors. In such circumstances to perform signal processing we need the area of interval-valued signal processing, more generally set-valued signal processing (see [1–3]). In this work, we introduce translation, modulation and dilation operators on $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ which is a special space of set-valued functions.

Unfortunately, the space $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ have an algebraic structure which is not a linear space. This structure is called as a “quasilinear space” by Aseev in 1986 [5]. Therefore, he present an approach for the function spaces of set-valued mappings. Let us give the definition of a quasilinear space which is presented by Aseev [5].

A set X is called a quasilinear space if a partial order relation “ \preceq ”, an algebraic sum operation, and an operation of multiplication by real numbers are defined in it in such a way that

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the following conditions hold for all elements $x, y, z, v \in X$ and all $\alpha, \beta \in \mathbb{R}$:

$$\begin{aligned}
& x \preceq x, \\
& x \preceq z \text{ if } x \preceq y \text{ and } y \preceq z, \\
& x = y \text{ if } x \preceq y \text{ and } y \preceq x, \\
& x + y = y + x, \\
& x + (y + z) = (x + y) + z, \\
& \text{there exists an element (zero) } \theta \in X \text{ such that } x + \theta = x, \\
& \alpha(\beta x) = (\alpha\beta)x, \\
& \alpha(x + y) = \alpha x + \alpha y, \\
& 1x = x, \\
& 0x = \theta, \\
& (\alpha + \beta)x \preceq \alpha x + \beta x, \\
& x + z \preceq y + v \text{ if } x \preceq y \text{ and } z \preceq v, \\
& \alpha x \preceq \alpha y \text{ if } x \preceq y.
\end{aligned}$$

Note that the concept of quasilinear space has been only introduced over the field \mathbb{R} . As distinct from Aseev's definition, in next section we will introduce the quasilinear spaces over general field \mathbb{K} which consists of real or complex numbers.

Any linear space is a quasilinear space with the partial order relation " $x \preceq y \iff x = y$ ".

Perhaps the most popular example of a nonlinear quasilinear space is the set of all nonempty closed intervals of real numbers sembolized by $\Omega_{\mathbb{C}}(\mathbb{R})$, and it is a quasilinear space with the inclusion relation " \subseteq ", the algebraic sum operation

$$A + B = \{a + b : a \in A, b \in B\}$$

and the real-scalar multiplication $\lambda A = \{\lambda a : a \in A\}$.

In fact $\Omega_{\mathbb{C}}(\mathbb{R})$ is the set of all nonempty compact convex subsets of real numbers and it is a subset of $\Omega(\mathbb{R})$, the set of all nonempty compact subsets of real numbers which is an another important example of a nonlinear quasilinear space. In general, $\Omega(E)$ and $\Omega_{\mathbb{C}}(E)$ are the sets of all nonempty closed bounded and nonempty convex closed bounded subsets of any normed linear space E , respectively. Both are a quasilinear space with the inclusion relation, the real-scalar multiplication and with a slight modification of addition as follows:

$$A + B = \overline{\{a + b : a \in A, b \in B\}},$$

where the closure is taken on the norm topology of E .

The investigation of $\Omega_{\mathbb{C}}(\mathbb{R})$ or more general $\Omega(\mathbb{C})$ contributes interval and convex analysis and they are excellent tools for mathematical formulation of many real-life situations, for example signal processing. Therefore we are interested in the space of $\Omega(\mathbb{C})$ -valued functions in this article.

We know the Banach space $L^p(\mathbb{R})$ for $1 \leq p < \infty$ the space of all functions f for which $|f|^p$ is integrable, is one of the fundamental vector spaces in functional analysis. In this paper we will try to investigate the space $L^p(\mathbb{R}, \Omega(\mathbb{C}))$ of all functions $F : \mathbb{R} \rightarrow \Omega(\mathbb{C})$ such that

$\int_{\mathbb{R}} \|F(x)\|_{\Omega}^p dx$ do exist ($1 \leq p < \infty$). We can see that the set $L^p(\mathbb{R}, \Omega(\mathbb{C}))$ is a normed quasilinear space and the special case $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ is a Hilbert quasilinear space. We use a new kind inner-product for set-valued functions to construct a norm structure of $L^2(\mathbb{R}, \Omega(\mathbb{C}))$. The inner-product will be introduced by an integral in the sense of Aumann [8].

1 PRELIMINARIES

We will start by giving the definition of quasilinear space which is different from Aseev's definition. In this definition we will consider the quasilinear spaces over a general field \mathbb{K} . The elements of \mathbb{K} are real or complex numbers. We think that this approach is suitable mathematical background of some applications, e.g., interval analysis and signal processing.

A set X is called a quasilinear space over field \mathbb{K} if a partial order relation " \preceq ", an algebraic sum operation, and an operation of multiplication by real or complex numbers are defined in it in such a way that the following conditions hold for any elements $x, y, z, v \in X$ and any $\alpha, \beta \in \mathbb{K}$:

$$\begin{aligned}
& x \preceq x, \\
& x \preceq z \text{ if } x \preceq y \text{ and } y \preceq z, \\
& x = y \text{ if } x \preceq y \text{ and } y \preceq x, \\
& x + y = y + x, \\
& x + (y + z) = (x + y) + z, \\
& \text{there exists an element } \theta \in X \text{ such that } x + \theta = x, \\
& \alpha(\beta x) = (\alpha\beta)x, \\
& \alpha(x + y) = \alpha x + \alpha y, \\
& 1x = x, \\
& 0x = \theta, \\
& (\alpha + \beta)x \preceq \alpha x + \beta x, \\
& x + z \preceq y + v \text{ if } x \preceq y \text{ and } z \preceq v, \\
& \alpha x \preceq \alpha y \text{ if } x \preceq y.
\end{aligned} \tag{1}$$

\mathbb{K} is called the scalar field of the quasilinear space X , and X is called a real quasilinear space if $\mathbb{K} = \mathbb{R}$ and is called a complex quasilinear space if $\mathbb{K} = \mathbb{C}$. Mostly \mathbb{K} will be \mathbb{C} in this work.

Any real linear space is a quasilinear space with the partial order relation defined by " $x \preceq y$ if and only if $x = y$ ". In this case, quasilinear space axioms is the linear space axioms.

Lemma 1 ([5]). *Suppose that each element x in quasilinear space X has an inverse element $x' \in X$. Then the partial order in X is determined by equality, the distributivity conditions hold, and consequently X is a linear space.*

Hence in a real linear space, the equality is the only way to define a partial order such that conditions (1) hold.

It will be assumed in what follows that $-x = (-1) \cdot x$. Also, note that $-x$ may not be x' . Any element x in a quasilinear space is regular if and only if $x - x = \theta$, that is, if and only if $x' = -x$.

Now, let us record some basic necessary results from [5]. In a quasilinear space X , the element θ is minimal, i.e., $x = \theta$ if $x \preceq \theta$. An element x' is called *inverse* of $x \in X$ if $x + x' = \theta$. The inverse is unique whenever it exists. An element x possessing inverse is called *regular*, otherwise is called *singular*.

Definition 1 ([6]). *Suppose that X is a quasilinear space and $Y \subseteq X$. Then Y is called a subspace of X whenever Y is a quasilinear space with the same partial order on X .*

Theorem 1 ([6]). *Y is subspace of quasilinear space X if and only if for every $x, y \in Y$ and $\alpha, \beta \in \mathbb{K}$, $\alpha \cdot x + \beta \cdot y \in Y$.*

Proof of this theorem is quite similar to its classical linear algebraic analogue.

Let X be a quasilinear space and Y be a subspace of X . Suppose that each element x in Y has inverse element $x' \in Y$ then by Lemma 1 the partial order on Y is determined by the equality. In this case Y is a linear subspace of X . An element x in quasilinear space X is said to be *symmetric* if $-x = x$ and X_{sym} denotes the set of all symmetric elements. Also, X_r stands for the set of all regular elements of X while X_s stands for the sets of all singular elements and zero in X . Further, it can be easily shown that X_r , X_{sym} and X_s are subspaces of X . They are called *regular*, *symmetric* and *singular subspaces* of X , respectively. Furthermore, it isn't hard to prove that summation of a regular element with a singular element is a singular element and the regular subspace of X is a linear space while the singular one is nonlinear at all.

Example 1. In $\Omega_C(\mathbb{R})$,

$$\{\{0\}\} \cup \{[a, b] : a, b \in \mathbb{R} \text{ and } a < b\}$$

is the singular subspace of $\Omega_C(\mathbb{R})$. Further $\{\{a\} : a \in \mathbb{R}\}$ is the set of all degenerate intervals or the set of all singletons of \mathbb{R} constitutes the regular subspace X_r . It is a linear subspace of $\Omega_C(\mathbb{R})$ and $(\Omega_C(\mathbb{R}))_r$ is the copy of \mathbb{R} in $\Omega_C(\mathbb{R})$. In fact, for any normed linear space E , each singleton $\{a\}$, $a \in E$, can be identified with the element a and hence E can be considered as the (regular) subspace of both $\Omega_C(E)$ and $\Omega(E)$. Further, the regular subspace of both $\Omega_C(E)$ and $\Omega(E)$ is isometrically isomorphic to E , namely, $(\Omega_C(E))_r \equiv E$ and $(\Omega(E))_r \equiv E$.

Let X be a real or complex quasilinear space. The real-valued function on X is called a *norm* if the following conditions hold:

$$\begin{aligned} & \|x\| > 0 \text{ if } x \neq 0, \\ & \|x + y\| \leq \|x\| + \|y\|, \\ & \|\alpha x\| = |\alpha| \|x\|, \\ & \text{if } x \preceq y, \text{ then } \|x\| \leq \|y\|, \\ & \text{if for any } \varepsilon > 0 \text{ there exists an element } x_\varepsilon \in X \text{ such that} \\ & x \preceq y + x_\varepsilon \text{ and } \|x_\varepsilon\| \leq \varepsilon \text{ then } x \preceq y, \end{aligned}$$

here x, y, x_ε are arbitrary element in X and α is any scalar.

A quasilinear space X with a norm defined on it, is called *normed quasilinear space*. It follows from Lemma 1 that if any $x \in X$ has inverse element $x' \in X$, then the concept of normed quasilinear space coincides with the concept of real normed linear space. Notice again that x'

may not exist but if x' exists then $x' = -x$. Hausdorff metric or norm metric on X is defined by the equality

$$h(x, y) = \inf \left\{ r \geq 0 : x \preceq y + a_1^{(r)}, y \preceq x + a_2^{(r)} \text{ and } \|a_i^{(r)}\| \leq r, i = 1, 2 \right\}.$$

Since $x \preceq y + (x - y)$ and $y \preceq x + (y - x)$, the quantity $h(x, y)$ is well-defined for any elements $x, y \in X$, and it is not hard to see that the function h satisfies all the metric axioms. Also we should note that $h(x, y)$ may not equal to $\|x - y\|$ if X is not a linear space; however $h(x, y) \leq \|x - y\|$ for every $x, y \in X$.

Lemma 2 ([5]). *The operations of algebraic sum and multiplication by real or complex numbers are continuous with respect to the Hausdorff metric. The norm is continuous with respect to the Hausdorff metric.*

Example 2 ([5]). *For a normed linear space E , a norm on $\Omega(E)$ is defined by*

$$\|A\|_{\Omega} = \sup_{a \in E} \|a\|_E.$$

Hence $\Omega_{\mathbb{C}}(E)$ and $\Omega(E)$ are normed quasilinear spaces. In this case the Hausdorff (norm) metric is defined as usual:

$$h(x, y) = \inf \{ r \geq 0 : x \subseteq y + S_r(\theta), y \subseteq x + S_r(\theta) \},$$

where $S_r(\theta)$ is a closed ball of E and x, y are elements of $\Omega_{\mathbb{C}}(E)$ or $\Omega(E)$. Further, $\Omega_{\mathbb{C}}(E)$ is a closed subspace of $\Omega(E)$.

Definition 2 ([5]). *A normed quasilinear space X is called an Ω -space if there exists an element $B_X \neq \theta$ such that*

$$\text{if } \|x\|_X \leq \|B_X\|_X, \text{ then } x \preceq B_X.$$

If X is a real normed linear space, then $\Omega(X)$ is an Ω -space.

Now, let us give a useful type of quasilinear spaces called *consolidate* quasilinear space.

Definition 3 ([6]). *Let X be a quasilinear space, $M \subseteq X$ and $x \in M$. The set*

$$F_x^M = \{z \in M_r : z \preceq x\}$$

is called floor in M of x . In the case of $M = X$ it is called only floor of x and written briefly F_x instead of F_x^X .

Floor of an element x in linear spaces is the singleton $\{x\}$. Therefore, it is nothing to discuss the notion of floor of an element in a linear space.

Definition 4 ([6]). *A quasilinear space X is called consolidate quasilinear space whenever $\sup F_y$ do exists for every $y \in X$ and*

$$y = \sup F_y = \sup \{z \in X_r : z \preceq y\}.$$

Otherwise, X is called non-consolidate quasilinear space.

Especially, we should note that the supremum in this definition is defined according to the partial order relation “ \preceq ” on X . Hence, we will use the notion of “sup” in place of general “ \sup ” to emphasize this case.

Example 3 ([6]). For any normed linear space E , $\Omega(E)$ and $\Omega_C(E)$ are consolidate normed quasilinear space.

Aseev launched a theory in [5] that we see it as the beginning of quasilinear functional analysis. However, there was a lot of deficiencies in the theory. One of them is the definition of inner-product. Now we will give the definition of inner-product in a quasilinear space which coincides with its linear analogue [6,7]. Later we will present some fundamental properties of inner-product and Hilbert quasilinear spaces. Firstly, let us introduce a definition.

Definition 5. For two quasilinear spaces (X, \preceq) and (Y, \preceq) , Y is called compatible contains X whenever $X \subseteq Y$ and the partial order relation \preceq on X is the restriction of the partial order relation \preceq on Y . We briefly use the symbol $X \subseteq Y$ in this case. We write $X \approx Y$ whenever $X \subseteq Y$ and $Y \subseteq X$.

Remark 1. $X \approx Y$ means X and Y are the same sets with the same partial order relations which make them quasilinear spaces. However, we may write $X = Y$ for $X \approx Y$ whenever the relations are clear from context.

Definition 6. Let X be a quasilinear space. Consolidation of X is the smallest consolidate quasilinear space \widehat{X} which compatible contains X , that is, if there exists another consolidate quasilinear space Y which compatible contains X then $\widehat{X} \subseteq Y$.

Clearly, $\widehat{X} = X$ for some consolidate quasilinear space X . We do not know yet whether each quasilinear space has a consolidation. This notion is unnecessary in consolidate quasilinear spaces, hence it is redundant in linear spaces. Further, $\widehat{\Omega_C(\mathbb{R}^n)} = \Omega_C(\mathbb{R}^n)$.

For a quasilinear space X , the set $F_y^{\widehat{X}} = \{z \in (\widehat{X})_r : z \preceq y\}$ is the floor of y in \widehat{X} .

Now, let us give an extended definition of inner-product given in [7]. We can say that the inner product in the following definition may be seen a set-valued inner product on quasilinear spaces.

Definition 7. Let X be a quasilinear space having a consolidation \widehat{X} . A mapping $\langle , \rangle : X \times X \rightarrow \Omega(\mathbb{K})$ is called an inner-product on X if for any $x, y, z \in X$ and $\alpha \in \mathbb{K}$ the following conditions are satisfied :

- If $x, y \in X_r$ then $\langle x, y \rangle \in \Omega_C(\mathbb{K})_r \equiv \mathbb{K}$,
- $\langle x + y, z \rangle \subseteq \langle x, z \rangle + \langle y, z \rangle$,
- $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$ and $\langle x, \alpha y \rangle = \bar{\alpha} \langle x, y \rangle$,
- $\langle x, y \rangle = \langle y, x \rangle$,
- $\langle x, x \rangle \geq 0$ for $x \in X_r$ and $\langle x, x \rangle = 0 \Leftrightarrow x = 0$,
- $\|\langle x, y \rangle\|_{\Omega} = \sup \left\{ \|\langle a, b \rangle\|_{\Omega} : a \in F_x^{\widehat{X}}, b \in F_y^{\widehat{X}} \right\}$,
- if $x \preceq y$ and $u \preceq v$ then $\langle x, u \rangle \subseteq \langle y, v \rangle$,
- if for any $\varepsilon > 0$ there exists an element $x_{\varepsilon} \in X$ such that $x \preceq y + x_{\varepsilon}$ and $\langle x_{\varepsilon}, x_{\varepsilon} \rangle \subseteq S_{\varepsilon}(\theta)$ then $x \preceq y$.

A quasilinear space with an inner product is called an *inner product quasilinear space*.

Remark 2. For some $x \in X$, $\langle x, x \rangle \geq 0$ means $\langle x, x \rangle$ is non-negative, that is, the order " \geq " in the definition is the usual order on $\Omega_{\mathbb{C}}(\mathbb{K})_r \equiv \mathbb{K}$. It should not be confused with the order " \preceq " on X .

Example 4 ([6,7]). Let X be a linear Hilbert space. Then the space $\Omega(X)$ is a Hilbert quasilinear space by the inner product defined by

$$\langle A, B \rangle_{\Omega} = \overline{\{\langle a, b \rangle_X : a \in A, b \in B\}}$$

for $A, B \in \Omega(X)$. Further, there is no need the closure for the definition of inner product on $\Omega(\mathbb{C})$, since $\{\langle a, b \rangle_{\mathbb{C}} : a \in A, b \in B\}$ is closed subset of \mathbb{C} . Namely, the inner product on $\Omega(\mathbb{C})$ is given by

$$\langle A, B \rangle_{\Omega} = \{\langle a, b \rangle_{\mathbb{C}} : a \in A, b \in B\}.$$

Every inner product quasilinear space X is a normed quasilinear space with the norm defined by

$$\|x\| = \sqrt{\|\langle x, x \rangle\|_{\Omega}}$$

for every $x \in X$. This norm is called *inner product norm*. Further $x_n \rightarrow x$ and $y_n \rightarrow y$ in a inner product quasilinear space then $\langle x_n, y_n \rangle \rightarrow \langle x, y \rangle$.

Lemma 3 ([6]). Let X be a inner product quasilinear space. Then

$$\|\langle x, y \rangle\|_{\Omega} \leq \|x\|_X \|y\|_X$$

for $x, y \in X$.

A inner product quasilinear space is called *Hilbert quasilinear space* if it is complete according to the inner-product (norm) metric. For example, $\Omega(\mathbb{C})$ is a Hilbert quasilinear space.

Definition 8 ([5]). Let X and Y be quasilinear spaces. A mapping $T : X \rightarrow Y$ is called a quasilinear operator if it satisfies the following conditions:

$$\begin{aligned} T(x_1 + x_2) &\preceq T(x_1) + T(x_2), \\ T(\alpha x) &= \alpha T(x) \text{ for any } \alpha \in \mathbb{R}, \\ \text{if } x_1 &\preceq x_2, \text{ then } T(x_1) \preceq T(x_2). \end{aligned}$$

Definition 9. Let X and Y be quasilinear spaces. A mapping $T : X \rightarrow Y$ is called a linear operator if it satisfies the following conditions:

$$\begin{aligned} T(x_1 + x_2) &= T(x_1) + T(x_2), \\ T(\alpha x) &= \alpha T(x) \text{ for any } \alpha \in \mathbb{R}, \\ \text{if } x_1 &\preceq x_2, \text{ then } T(x_1) \preceq T(x_2). \end{aligned}$$

Hence linear operators can be obtained by adding an extra condition to the first condition of quasilinear operators.

Remark 3. We will see that quasilinear operators may not conserve quasilinear structure. Due to this obstacle we introduce the linear operator notion acting on quasilinear spaces. Obviously, any linear operator between quasilinear spaces is a quasilinear operator, but not conversely. If X and Y are linear spaces then the definition of quasilinear operators coincides with the usual definition of a linear operators.

Definition 10 ([5]). Let X and Y be a normed quasilinear spaces. A quasilinear operator $T : X \rightarrow Y$ is said to be bounded if there exists a number $k > 0$ such that $\|Tx\| \leq k \|x\|$ for any $x \in X$.

2 AUMANN INTEGRAL

We will need the integral of set-valued functions when we deal with the space $L^p(\mathbb{R}, \Omega(\mathbb{C}))$, $1 \leq p < \infty$. For this purpose we will introduce the integral of a set-valued function and give some properties of this integral.

Integrals of set-valued functions are given by Robert J. Aumann in 1965. It is as follows [8]:

Let I be the unit interval $[0, 1]$. For any t in I , let $F(t)$ be a nonempty subset of \mathbb{R}^n . Suppose that \mathcal{L} be the set of all point-valued functions f from I to \mathbb{R}^n such that f is integrable over I and $f(t) \in F(t)$ for all t in I . Define

$$\int_I F(t)dt = \left\{ \int_I f(t)dt : f \in \mathcal{L} \right\}$$

i.e., the set of all integrals of members of \mathcal{L} .

Throughout the section we will use the notations: The triple $(\Gamma, \mathcal{A}, \mu)$ is a complete σ -finite measure space, X is a complete separable metric space and $F : \Gamma \rightsquigarrow X$ represents a set-valued function that assigns to each $t \in \Gamma$ a subset $F(t) \subseteq X$.

Let us give the main definitions and theorems with respect to the integral of a measurable set-valued function.

Definition 11 ([9]). A set-valued function $F : \Gamma \rightsquigarrow X$ is called with closed, open or compact valued if $F(x)$ is a closed, open or compact set in X , for each $x \in \Gamma$, respectively.

Definition 12 ([9]). A set-valued function $F : \Gamma \rightsquigarrow X$ is called measurable if for any open subset $\mathcal{O} \subset X$,

$$F^{-1}(\mathcal{O}) = \{x \in \Gamma : F(x) \cap \mathcal{O} \neq \emptyset\}$$

is element of \mathcal{A} .

Measurability of set-valued functions is closely associated with the concept of measurability of its selections.

Definition 13 ([9]). For a given set-valued function $F : \Gamma \rightsquigarrow X$, a measurable function $f : \Gamma \rightarrow X$ satisfying

$$\text{for all } x \in \Gamma, f(x) \in F(x)$$

is called a measurable selection of F .

Theorem 2 ([9]). *Let $F : \Gamma \rightsquigarrow X$ be closed valued. Then there exists a measurable selection of F .*

We denote by $L^p(\Gamma, X, \mu)$, $1 \leq p < \infty$ the Banach space of all measurable functions $f : \Gamma \rightarrow X$ such that $\int_{\Gamma} \|f\|^p d\mu < \infty$. If $\Gamma = \mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and μ is the Lebesgue measure then we find $L^p(\Gamma, X, \mu) = L^p(\mathbb{R})$. For $1 \leq p < \infty$, $S^p(F)$ is the set of all selections $f \in L^p(\Gamma, X, \mu)$ of a measurable set-valued function $F : \Gamma \rightsquigarrow X$ [10], i.e.,

$$S^p(F) = \left\{ f : \Gamma \rightarrow X : \int_{\Gamma} \|f\|^p d\mu < \infty, \text{ and } f(x) \in F(x) \text{ for } x \in \Gamma \right\}.$$

Definition 14 ([4]). *A set-valued function $F : \Gamma \rightsquigarrow X$ is called integrably bounded if there exists a nonnegative function $f \in L^1(\Gamma, \mathbb{R}, \mu)$ such that*

$$F(x) \subset f(x)B \text{ almost everywhere in } \Gamma,$$

where B is the unit ball of X .

Aumann gave the definition of an integral of a set-valued function in the following way:

Definition 15 ([4]). *The integral of F on Γ is the set of integrals of integrable selections of F :*

$$\int_{\Gamma} F d\mu = \left\{ \int_{\Gamma} f d\mu : f \in S^1(F) \right\}.$$

We will say that F is integrable set-valued function in the sense of Aumann if the set $\left\{ \int_{\Gamma} f d\mu : f \in S^1(F) \right\}$ is not empty. Aumann integral of F will be shown as $\int_{\Gamma}^{(A)} F d\mu$.

Proposition 1 ([9]). *If $G : \Gamma \rightsquigarrow X$ is Aumann integrable and $G(x) \subseteq F(x)$ almost everywhere on Γ . Then the set-valued function F is also Aumann integrable and*

$$\int_{\Gamma}^{(A)} G(x) dx \subseteq \int_{\Gamma}^{(A)} F(x) dx.$$

Proposition 2 ([9]). *If $F, F_1, F_2 : \Gamma \rightsquigarrow X$ are Aumann integrable then $F_1 + F_2$ and λF are Aumann integrable and*

$$\int_{\Gamma}^{(A)} (F_1 + F_2)(x) dx = \int_{\Gamma}^{(A)} F_1(x) dx + \int_{\Gamma}^{(A)} F_2(x) dx$$

and

$$\int_{\Gamma}^{(A)} (\lambda F)(x) dx = \lambda \int_{\Gamma}^{(A)} F(x) dx.$$

Proposition 3 ([9]). *If $F : \mathbb{R} \rightarrow \Omega(X)$ is Aumann integrable and the integral of F is compact then*

$$\left\| \int_{\Gamma}^{(A)} F(x) dx \right\|_{\Omega} \leq \int_{\Gamma}^{(A)} \|F(x)\|_{\Omega} dx.$$

Theorem 3 ([4]). *Let $F : \Gamma \rightsquigarrow \mathbb{R}^n$ be a measurable and closed-valued function. If μ is nonatomic and F is integrably bounded, then the Aumann integral of F is compact.*

Now let us present the *Dominated Convergence Theorem* for the Aumann integrals.

Theorem 4 ([11]). *If $F_n : \Gamma \rightarrow \Omega(\mathbb{C})$ $n = 1, 2, \dots$ are measurable closed valued functions, $\{\|F_n(\cdot)\|\}_{n=1}^\infty$ is uniformly integrable and $F_n(x) \rightarrow F(x)$ with respect to the Hausdorff metric then*

$$\overline{\int_{\Gamma}^{(A)} F_n(x) dx} \rightarrow \overline{\int_{\Gamma}^{(A)} F(x) dx}.$$

3 THE HILBERT QUASILINEAR SPACE $L^2(\mathbb{R}, \Omega(\mathbb{C}))$

In this chapter we will concentrate on the quasilinear structure of the $L^p(\mathbb{R}, \Omega(\mathbb{C}))$ space, $1 \leq p < \infty$. We will show that $L^p(\mathbb{R}, \Omega(\mathbb{C}))$ spaces are normed quasilinear space over the field \mathbb{C} and later we construct a set-valued inner-product on $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ by way of Aumann integral.

For $1 \leq p < \infty$, the space $L^p(\mathbb{R}, \Omega(\mathbb{C}))$ consists of all set-valued measurable functions $F : \mathbb{R} \rightarrow \Omega(\mathbb{C})$ such that the Lebesgue integral

$$\int_{\mathbb{R}} \|F(x)\|_{\Omega}^p dx$$

is well defined, where the notion of measurability of F is the measurability in Definition 12. Note that this integral is a classical Lebesgue integral.

Among the $L^p(\mathbb{R}, \Omega(\mathbb{C}))$ spaces, the case $p = 2$ has a special importance: We will say that $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ is an inner-product quasilinear space with respect to the inner-product which is defined via Aumann integral

$$\langle F, G \rangle = \int_{\mathbb{R}}^{(A)} \langle F(x), G(x) \rangle_{\Omega} dx. \quad (2)$$

Firstly, let us indicate $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ is a consolidate quasilinear space and so it has a consolidation. Therefore, we can define a set-valued inner-product function on this space. After the definition of inner-product on $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ we will denote the norm on $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ with

$$\|F\| = \left\| \left\{ \int_{\mathbb{R}} |f(x)|^2 dx : f \in S^2(F) \right\} \right\|_{\Omega}^{1/2}$$

and we will show that this norm comes from the inner-product given by the equality (2). Thereafter, we will show that the inner-product norm on $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ coincides with the expression

$$\left(\int_{\mathbb{R}} \|F(x)\|_{\Omega}^2 dx \right)^{1/2},$$

namely, the equality

$$\|F\| = \left(\int_{\mathbb{R}} \|F(x)\|_{\Omega}^2 dx \right)^{1/2}$$

is also a norm on $L^2(\mathbb{R}, \Omega(\mathbb{C}))$. Further, we will prove that $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ is a Banach quasilinear space with this norm. Thus, we will say that $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ is a Hilbert quasilinear space.

The operations of algebraic sum, multiplication by a complex scalar and the partial order relation are defined as follows:

$$(F_1 + F_2)(x) = F_1(x) + F_2(x), \quad (\lambda F)(x) = \lambda F(x)$$

and

$$F_1 \preceq F_2 \Leftrightarrow F_1(x) \subseteq F_2(x) \text{ for any } x \in \mathbb{R}.$$

By a similar way given in [5], it is easy to verify that $L^p(\mathbb{R}, \Omega(\mathbb{C}))$ is a quasilinear space over the field \mathbb{C} by the above algebraic operations and the relation.

Now let us determine the regular elements of $L^p(\mathbb{R}, \Omega(\mathbb{C}))$, $1 \leq p < \infty$:

$$\begin{aligned} F \in (L^p(\mathbb{R}, \Omega(\mathbb{C})))_r &\Leftrightarrow F - F = \theta \Leftrightarrow F(x) - F(x) = \{0\}, \text{ for all } x \in \mathbb{R} \\ &\Leftrightarrow F(x) \in \Omega(\mathbb{C})_r \equiv \mathbb{C}, \text{ for all } x \in \mathbb{R}. \end{aligned}$$

By $\Omega(\mathbb{C})_r \equiv \mathbb{C}$ we mean there exist an isometric isomorphism (equivalence) between these normed linear spaces. Recall again that the regular subspace of a quasilinear space is just a linear space. Hence we can give the following corollaries.

Corollary 1. $(L^p(\mathbb{R}, \Omega(\mathbb{C})))_r = L^p(\mathbb{R}, \Omega(\mathbb{C})_r) \equiv L^p(\mathbb{R}, \mathbb{C}) = L^p(\mathbb{R})$ for $1 \leq p < \infty$. Further, if $F \in L^p((\mathbb{R}, \Omega(\mathbb{C})))_r$ then there exists only one selection of F and this selection is equal to itself.

Now we will prove that the $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ space is a inner-product quasilinear space.

Theorem 5. *The quasilinear space $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ is an inner-product quasilinear space with respect to the inner-product*

$$\langle F, G \rangle = \int_{\mathbb{R}} \langle F(x), G(x) \rangle_{\Omega} dx \quad (A) \quad (3)$$

for $F, G \in L^2(\mathbb{R}, \Omega(\mathbb{C}))$ and using the Aumann integral gives the equality

$$\langle F, G \rangle = \int_{\mathbb{R}} \langle F(x), G(x) \rangle_{\Omega} dx = \left\{ \int_{\mathbb{R}} \langle f(x), g(x) \rangle_{\mathbb{C}} dx : f \in S^2(F), g \in S^2(G) \right\}. \quad (4)$$

Proof. Previously, we shall verify that the equality (3) is well-defined, i.e., that the function

$$U_{F,G} : \mathbb{R} \rightarrow \Omega(\mathbb{C}), \quad U_{F,G}(x) = \langle F(x), G(x) \rangle_{\Omega}$$

is integrable according to Aumann and this integral belongs to $\Omega(\mathbb{C})$ (see, Definition 7). If we consider the Theorem 2 then we can say that $U_{F,G}$ has a measurable selection, since $U_{F,G}$ is closed valued. Thus, this function is integrable according to Aumann. Now we will show

that the Aumann integral of $U_{F,G}$ is an element of $\Omega(\mathbb{C})$: Firstly, let us show that $U_{F,G}$ is integrably bounded, namely, there exists a nonnegative integrable function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $U_{F,G}(x) \subseteq f(x)B$ for any $x \in \mathbb{R}$, where $B = \{a \in \mathbb{C} : |a| \leq 1\}$. By the definition of norm on $\Omega(\mathbb{C})$,

$$\|U_{F,G}(x)\|_{\Omega} = \|\langle F(x), G(x) \rangle_{\Omega}\| = \sup\{|\langle a_x, b_x \rangle_{\mathbb{C}}| : a_x \in F(x), b_x \in G(x)\}.$$

Since for each $x \in \mathbb{R}$, $U_{F,G}(x)$ is a compact set, there exists the elements $a_x^0 \in F(x)$ and $b_x^0 \in G(x)$ which are dependent on x such that

$$\|U_{F,G}(x)\|_{\Omega} = \|\langle F(x), G(x) \rangle_{\Omega}\| = \left| \langle a_x^0, b_x^0 \rangle_{\mathbb{C}} \right|.$$

By reason of the fact that each of the elements x corresponds to the element $a_x^0 \in F(x)$ and $b_x^0 \in G(x)$, we can define the function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x) = \left| \langle a_x^0, b_x^0 \rangle_{\mathbb{C}} \right|.$$

Further,

$$\left| \int_{\mathbb{R}} f(x) dx \right| \leq \int_{\mathbb{R}} |f(x)| dx = \int_{\mathbb{R}} \left| \langle a_x^0, b_x^0 \rangle_{\mathbb{C}} \right| dx = \int_{\mathbb{R}} \|U_{F,G}(x)\|_{\Omega} dx = \int_{\mathbb{R}} \|\langle F(x), G(x) \rangle_{\Omega}\| dx.$$

By Lemma 3 and Holder inequality we observe that

$$\int_{\mathbb{R}} \|\langle F(x), G(x) \rangle_{\Omega}\| dx \leq \int_{\mathbb{R}} (\|F(x)\|_{\Omega} \|G(x)\|_{\Omega}) dx \leq \left(\int_{\mathbb{R}} \|F(x)\|_{\Omega}^2 dx \right)^{1/2} \left(\int_{\mathbb{R}} \|G(x)\|_{\Omega}^2 dx \right)^{1/2}.$$

The last inequality implies f is integrable since $F, G \in L^2(\mathbb{R}, \Omega(\mathbb{C}))$. Furthermore,

$$\|U_{F,G}(x)\|_{\Omega} = \left| \langle a_x^0, b_x^0 \rangle_{\mathbb{C}} \right| = \left| \langle a_x^0, b_x^0 \rangle_{\mathbb{C}} \right| \|B\| = \|f(x)B\|.$$

Since $\Omega(\mathbb{C})$ is an Ω -space, we have that $U_{F,G}(x) \subseteq f(x)B$ for any $x \in \mathbb{R}$ and so $U_{F,G}$ is integrably bounded. Consequently, by the Theorem 3 we say that the Aumann integral of $U_{F,G}$

$$\langle F, G \rangle = \int_{\mathbb{R}}^{(A)} U_{F,G}(x) dx = \int_{\mathbb{R}}^{(A)} \langle F(x), G(x) \rangle_{\Omega} dx$$

is a compact set. The next step is to verify the equality (4): If we apply the definition of Aumann integral to the set-valued function $U_{F,G}$ then we write

$$\langle F, G \rangle = \int_{\mathbb{R}}^{(A)} \langle F(x), G(x) \rangle_{\Omega} dx = \int_{\mathbb{R}}^{(A)} U_{F,G}(x) dx = \left\{ \int_{\mathbb{R}} h(x) dx : h \in S(U_{F,G}) \right\}.$$

Now let us research the selections of $U_{F,G}$. By the definition of norm on $\Omega(\mathbb{C})$ we write

$$\langle F(x), G(x) \rangle_{\Omega} = \{ \langle z, w \rangle_{\mathbb{C}} : z \in F(x), w \in G(x) \}.$$

If we remember that $h(x) \in U_{F,G}(x)$ for every $x \in \mathbb{R}$ then for the determined elements $z_x^0 \in F(x)$ and $w_x^0 \in G(x)$ it is written that

$$h(x) = \langle z_x^0, w_x^0 \rangle_{\mathbb{C}},$$

where z_x^0 and w_x^0 are depend on the element x . Let us describe the functions $f : \mathbb{R} \rightarrow \mathbb{C}$ and $g : \mathbb{R} \rightarrow \mathbb{C}$ such that $f(x) = z_x^0$ and $g(x) = w_x^0$. The functions f and g are well-defined due to the fact that h is a function. It is obvious that $f \in S^2(F)$ and $g \in S^2(G)$ and so $f, g \in L^2(\mathbb{R})$. Also we can see that

$$h(x) = \langle f(x), g(x) \rangle_{\mathbb{C}}$$

for any element x . The equality

$$\left| \int_{\mathbb{R}} h(x) dx \right| = \left| \int_{\mathbb{R}} \langle f(x), g(x) \rangle_{\mathbb{C}} dx \right| \leq \int_{\mathbb{R}} |\langle f(x), g(x) \rangle_{\mathbb{C}}| dx = \int_{\mathbb{R}} |f(x) \overline{g(x)}| dx$$

and from the Cauchy-Shwarz inequality give

$$\int_{\mathbb{R}} |f(x) \overline{g(x)}| dx \leq \left(\int_{\mathbb{R}} |f(x)|^2 dx \right)^{1/2} \left(\int_{\mathbb{R}} |g(x)|^2 dx \right)^{1/2} < \infty$$

and so $h \in S(U_{F,G})$. Hence,

$$\langle F, G \rangle = \int_{\mathbb{R}} \langle F(x), G(x) \rangle_{\Omega} dx \stackrel{(A)}{=} \left\{ \int_{\mathbb{R}} \langle f(x), g(x) \rangle_{\mathbb{C}} dx : f \in S^2(F), g \in S^2(G) \right\}.$$

Now we shall show that the expression (3) defines an inner product on $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ in the meaning of the Definition 7.

1. If $F, G \in L^2(\mathbb{R}, \Omega(\mathbb{C}))$ then $\langle F, G \rangle \in \Omega(\mathbb{C})_r \cong \mathbb{C}$.

If $F, G \in L^2(\mathbb{R}, \Omega(\mathbb{C}))$ then by the Corollary 1

$$\langle F, G \rangle = \int_{\mathbb{R}} \langle F(x), G(x) \rangle_{\Omega} dx \stackrel{(A)}{=} \int_{\mathbb{R}} \langle F(x), G(x) \rangle_{\mathbb{C}} dx = \int_{\mathbb{R}} F(x) \overline{G(x)} dx.$$

Also if we remember that the equality

$$\langle F, G \rangle = \int_{\mathbb{R}} F(x) \overline{G(x)} dx$$

is complex-valued inner product on $L^2(\mathbb{R})$ then we say that $\langle F, G \rangle \in \Omega(\mathbb{C})_r \cong \mathbb{C}$.

2. $\langle F + G, H \rangle = \langle F, H \rangle + \langle G, H \rangle$:

By the second condition of inner product on $\Omega(\mathbb{C})$ and the Proposition 1 we have that

$$\langle F + G, H \rangle = \int_{\mathbb{R}} \langle F(x) + G(x), H(x) \rangle_{\Omega} dx \stackrel{(A)}{\subseteq} \int_{\mathbb{R}} (\langle F(x), H(x) \rangle_{\Omega} + \langle G(x), H(x) \rangle_{\Omega}) dx \stackrel{(A)}{=}$$

and from the Proposition 2 we obtain that

$$\langle F + G, H \rangle \stackrel{(A)}{\subseteq} \int_{\mathbb{R}} \langle F(x), H(x) \rangle_{\Omega} dx + \int_{\mathbb{R}} \langle G(x), H(x) \rangle_{\Omega} dx = \langle F, H \rangle + \langle G, H \rangle.$$

3. $\langle \lambda F, G \rangle = \lambda \langle F, G \rangle$ and $\langle F, \lambda G \rangle = \bar{\lambda} \langle F, G \rangle$:

By the third condition of inner product on $\Omega(\mathbb{C})$ and the Proposition 2 we have that

$$\begin{aligned} \langle \lambda F, G \rangle &= \int_{\mathbb{R}}^{(A)} \langle (\lambda F)(x), G(x) \rangle_{\Omega} dx = \int_{\mathbb{R}}^{(A)} \lambda \langle F(x), G(x) \rangle_{\Omega} dx \\ &= \lambda \int_{\mathbb{R}}^{(A)} \langle F(x), G(x) \rangle_{\Omega} dx = \lambda \langle F, G \rangle. \end{aligned}$$

It can be easily shown that $\langle F, \lambda G \rangle = \bar{\lambda} \langle F, G \rangle$.

4. $\langle F, G \rangle = \langle G, F \rangle$:

By the fourth condition of inner product on $\Omega(\mathbb{C})$,

$$\langle F, G \rangle = \int_{\mathbb{R}}^{(A)} \langle F(x), G(x) \rangle_{\Omega} dx = \int_{\mathbb{R}}^{(A)} \langle G(x), F(x) \rangle_{\Omega} dx = \langle G, F \rangle.$$

5. $\langle F, F \rangle \geq 0$ for $F \in (L^2(\mathbb{R}, \Omega(\mathbb{C})))_r$ and $\langle F, F \rangle = \{0\} \Leftrightarrow F = \theta$:

If $F \in (L^2(\mathbb{R}, \Omega(\mathbb{C})))_r$, then $f \in L^2(\mathbb{R})$ by the Corollary (1) and so

$$\begin{aligned} \langle F, F \rangle &= \int_{\mathbb{R}}^{(A)} \langle F(x), F(x) \rangle_{\Omega} dx = \left\{ \int_{\mathbb{R}} \langle F(x), F(x) \rangle_{\mathbb{C}} dx \right\} \\ &= \left\{ \int_{\mathbb{R}} F(x) \overline{F(x)} dx \right\} = \left\{ \int_{\mathbb{R}} |F(x)|^2 dx \right\}. \end{aligned}$$

Since the inner-product on $L^2(\mathbb{R})$ is non-negative we have that $\int_{\mathbb{R}} |F(x)|^2 dx \geq 0$ and so $\langle F, F \rangle \geq 0$.

Now let us assume that $\langle F, F \rangle = 0$. Then

$$\int_{\mathbb{R}}^{(A)} \langle F(x), F(x) \rangle_{\Omega} dx = \left\{ \int_{\mathbb{R}} \langle f(x), g(x) \rangle_{\mathbb{C}} dx : f, g \in S^2(F) \right\} = \{0\}.$$

This implies $\int_{\mathbb{R}} f(x) \overline{f(x)} dx = \int_{\mathbb{R}} |f(x)|^2 dx = 0$. Hence, by the Corollary (1) and the norm on $L^2(\mathbb{R})$ we say that $f = 0$. Since the any selection f of F is equal to 0, we say that $F = \theta$.

6. $\| \langle F, G \rangle \|_{\Omega} = \sup \{ \| \langle f, g \rangle \|_{\Omega} : f \in F_F, g \in F_G \}$:

Firstly, it is not hard to see that $F_F \subseteq S^2(F)$ for $F \in L^2(\mathbb{R}, \Omega(\mathbb{C}))$. By this way we say that

$$\begin{aligned} \sup \{ \| \langle f, g \rangle \|_{\Omega} : f \in F_F, g \in F_G \} &= \sup \left\{ \left| \int_{\mathbb{R}} \langle f(x), g(x) \rangle_{\mathbb{C}} dx \right| : f \in F_F, g \in F_G \right\} \\ &= \sup \left\{ \left| \int_{\mathbb{R}} \langle f(x), g(x) \rangle_{\mathbb{C}} dx \right| : f \in S^2(F), g \in S^2(G) \right\} = \| \langle F, G \rangle \|_{\Omega}. \end{aligned}$$

7. $\langle F_1, G_1 \rangle \subseteq \langle F_2, G_2 \rangle$ if $F_1 \preceq F_2$ and $G_1 \preceq G_2$:

If $F_1 \preceq F_2$ and $G_1 \preceq G_2$ then $F_1(x) \subseteq F_2(x)$ and $G_1(x) \subseteq G_2(x)$ for a.e. $x \in \mathbb{R}$. By the seventh condition of inner product on $\Omega(\mathbb{C})$ we say that

$$\langle F_1(x), G_1(x) \rangle \subseteq \langle F_2(x), G_2(x) \rangle .$$

Therefore, using the Proposition 1 implies the equality

$$\langle F_1, G_1 \rangle = \int_{\mathbb{R}}^{(A)} \langle F_1(x), G_1(x) \rangle dx \subseteq \int_{\mathbb{R}}^{(A)} \langle F_2(x), G_2(x) \rangle dx = \langle F_2, G_2 \rangle .$$

8. We show that if for any $\varepsilon > 0$ there exists an element $F_\varepsilon \in L^2(\mathbb{R}, \Omega(\mathbb{C}))$ such that $F \preceq G + F_\varepsilon$ and $\langle F_\varepsilon, F_\varepsilon \rangle \subseteq S_\varepsilon(\theta)$ then $F \preceq G$:

Suppose that for any $\varepsilon > 0$ there exists an element $F_\varepsilon \in L^2(\mathbb{R}, \Omega(\mathbb{C}))$ such that $F \preceq G + F_\varepsilon$ and $\langle F_\varepsilon, F_\varepsilon \rangle \subseteq S_\varepsilon(\theta)$. Then

$$\|\langle F_\varepsilon, F_\varepsilon \rangle\|_\Omega \leq \|S_\varepsilon(\theta)\|_\Omega = \varepsilon. \quad (5)$$

Further,

$$\begin{aligned} \|\langle F_\varepsilon, F_\varepsilon \rangle\|_\Omega &= \left\| \int_{\mathbb{R}}^{(A)} \langle F(x), F(x) \rangle_\Omega dx \right\|_\Omega = \left\| \left\{ \int_{\mathbb{R}} \langle f_\varepsilon(x), g_\varepsilon(x) \rangle_{\mathbb{C}} dx : f_\varepsilon, g_\varepsilon \in S^2(F_\varepsilon) \right\} \right\|_\Omega \\ &= \sup \left\{ \left| \int_{\mathbb{R}} \langle f_\varepsilon(x), f_\varepsilon(x) \rangle_{\mathbb{C}} dx \right| : f_\varepsilon \in S^2(F_\varepsilon) \right\} \\ &= \sup \left\{ \left| \int_{\mathbb{R}} |f_\varepsilon(x)|^2 dx \right| : f_\varepsilon \in S^2(F_\varepsilon) \right\} = \|F_\varepsilon\|^2. \end{aligned}$$

Hence by the inequality (5) we say that $\|F_\varepsilon\|^2 \leq \varepsilon$. The last condition of norm on $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ indicates $F \preceq G$.

□

For $1 \leq p < \infty$, the expression

$$\|F\| = \left\| \left\{ \int_{\mathbb{R}} |f(x)|^p dx : f \in S^p(F) \right\} \right\|_\Omega^{1/p}$$

defines a norm on $L^p(\mathbb{R}, \Omega(\mathbb{C}))$ and so this space is a normed quasilinear space and this norm is an inner-product norm obtained from the inner-product (3). Notably,

$$\begin{aligned} \|F\|^2 &= \|\langle F, F \rangle\| = \left\| \int_{\mathbb{R}}^{(A)} \langle F(x), F(x) \rangle_\Omega dx \right\|_\Omega = \left\| \left\{ \int_{\mathbb{R}} \langle f(x), g(x) \rangle_{\mathbb{C}} dx : f, g \in S^2(F) \right\} \right\|_\Omega \\ &= \sup \left\{ \left| \int_{\mathbb{R}} \langle f(x), g(x) \rangle_{\mathbb{C}} dx \right| : f, g \in S^2(F) \right\} = \sup \left\{ \left| \int_{\mathbb{R}} \langle f(x), f(x) \rangle_{\mathbb{C}} dx \right| : f \in S^2(F) \right\} \\ &= \sup \left\{ \int_{\mathbb{R}} |f(x)|^2 dx : f \in S^2(F) \right\}. \end{aligned}$$

Since for bounded subset $A \subset \mathbb{C}$ we have that $\sup |A| = \sup |\overline{A}|$ where $|A| = \{|a| : a \in A\}$ then

$$\begin{aligned} \|F\|^2 &= \|\langle F, F \rangle\| = \sup \left\{ \int_{\mathbb{R}} |f(x)|^2 dx : f \in S^2(F) \right\} = \sup \overline{\left\{ \int_{\mathbb{R}} |f(x)|^2 dx : f \in S^2(F) \right\}} \\ &= \left\| \overline{\left\{ \int_{\mathbb{R}} |f(x)|^2 dx : f \in S^2(F) \right\}} \right\|_{\Omega} \end{aligned}$$

and so

$$\|F\| = \left\| \overline{\left\{ \int_{\mathbb{R}} |f(x)|^2 dx : f \in S^2(F) \right\}} \right\|_{\Omega}^{1/2}.$$

Lemma 4. For $p = 2$, the inner-product norm is equivalent to $(\int_{\mathbb{R}} \|F(x)\|_{\Omega}^2 dx)^{1/2}$ i.e., if $F \in L^2(\mathbb{R}, \Omega(\mathbb{C}))$ then

$$\|F\| = \left\| \overline{\left\{ \int_{\mathbb{R}} |f(x)|^2 dx : f \in S^2(F) \right\}} \right\|_{\Omega}^{1/2} = \left(\int_{\mathbb{R}} \|F(x)\|_{\Omega}^2 dx \right)^{1/2}. \quad (6)$$

Proof. By the Proposition 3 and the norm of inner-product on $\Omega(\mathbb{C})$, we write

$$\|F\|^2 = \|\langle F, F \rangle\| = \left\| \int_{\mathbb{R}} \langle F(x), F(x) \rangle_{\Omega} dx \right\| \leq \int_{\mathbb{R}} \|\langle F(x), F(x) \rangle_{\Omega}\| dx = \int_{\mathbb{R}} \|F(x)\|_{\Omega}^2 dx \quad (7)$$

and

$$\|F\|^2 = \|\langle F, F \rangle\| = \left\| \overline{\left\{ \int_{\mathbb{R}} |f(x)|^2 dx : f \in S^2(F) \right\}} \right\|_{\Omega}. \quad (8)$$

Using the (7) and (8) we have the inequality

$$\|F\|^2 = \|\langle F, F \rangle\| = \left\| \overline{\left\{ \int_{\mathbb{R}} |f(x)|^2 dx : f \in S^2(F) \right\}} \right\|_{\Omega} \leq \int_{\mathbb{R}} \|F(x)\|_{\Omega}^2 dx. \quad (9)$$

It is obvious that for any $x \in \mathbb{R}$, $F(x)$ is a compact subset of \mathbb{C} . Hence, there exists an element t_0^x in $F(x)$ such that

$$\sup\{|t| : t \in F(x)\} = |t_0^x|.$$

Let us define the function $g : \mathbb{R} \rightarrow \mathbb{C}$ with $g(x) = t_0^x$. It is not hard to see that g is well-defined. The function g is an element of $S^2(F)$ due to the fact that for $x \in \mathbb{R}$, $g(x) = t_0^x \in F(x)$ and

$$\int_{\mathbb{R}} \|F(x)\|_{\Omega}^2 dx = \int_{\mathbb{R}} (\sup\{|t| : t \in F(x)\})^2 dx = \int_{\mathbb{R}} |t_0^x|^2 dx = \int_{\mathbb{R}} |g(x)|^2 dx. \quad (10)$$

Since

$$\int_{\mathbb{R}} |g(x)|^2 dx \leq \sup \left\{ \int_{\mathbb{R}} |f(x)|^2 dx : f \in S^2(F) \right\}$$

and

$$\begin{aligned} \sup_{\mathbb{R}} \left\{ \int_{\mathbb{R}} |f(x)|^2 dx : f \in S^2(F) \right\} &= \sup_{\mathbb{R}} \overline{\left\{ \int_{\mathbb{R}} |f(x)|^2 dx : f \in S^2(F) \right\}} \\ &= \left\| \overline{\left\{ \int_{\mathbb{R}} |f(x)|^2 dx : f \in S^2(F) \right\}} \right\|_{\Omega}. \end{aligned}$$

This implies

$$\int_{\mathbb{R}} |g(x)|^2 dx \leq \left\| \overline{\left\{ \int_{\mathbb{R}} |f(x)|^2 dx : f \in S^2(F) \right\}} \right\|_{\Omega}. \quad (11)$$

By the (10) and (11) we say that

$$\int_{\mathbb{R}} \|F(x)\|_{\Omega}^2 dx = \left\| \overline{\left\{ \int_{\mathbb{R}} |f(x)|^2 dx : f \in S^2(F) \right\}} \right\|_{\Omega}. \quad (12)$$

Therefore, from the (9) and (12) we obtain that

$$\|F\| = \left\| \overline{\left\{ \int_{\mathbb{R}} |f(x)|^2 dx : f \in S^2(F) \right\}} \right\|_{\Omega}^{1/2} = \left(\int_{\mathbb{R}} \|F(x)\|_{\Omega}^2 dx \right)^{1/2}.$$

□

Theorem 6. *The quasilinear space $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ is complete with the norm given by (6), i.e., $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ is a Banach quasilinear space.*

Proof. Let $\{F_n\}_{n=1}^{\infty}$ be a sequence in $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ such that $\sum_{k=1}^{\infty} \|F_k\| < \infty$. We will show that the series $\sum_{k=1}^{\infty} F_k$ is convergent. For this we need to find a function F in $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ such that

$$\lim_{n \rightarrow \infty} h_{L^2} \left(\sum_{k=1}^n F_k, F \right) = 0$$

where h_{L^2} is the Hausdorff metric on the normed quasilinear space $L^2(\mathbb{R}, \Omega(\mathbb{C}))$. Now we define the function $g : \mathbb{R} \rightarrow \mathbb{R}$ by

$$g(x) = \sum_{k=1}^{\infty} (\|F_k(x)\|_{\Omega})^2.$$

Applying the Monoton Convergence Theorem and Minkowski inequality prove that

$$\begin{aligned} \int_{\mathbb{R}} g(x) dx &= \int_{\mathbb{R}} \left(\sum_{k=1}^{\infty} (\|F_k(x)\|_{\Omega})^2 \right) dx = \int_{\mathbb{R}} \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n (\|F_k(x)\|_{\Omega})^2 \right) dx \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}} \left(\sum_{k=1}^n (\|F_k(x)\|_{\Omega})^2 \right) dx \leq \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \left(\int_{\mathbb{R}} \|F_k(x)\|_{\Omega}^2 dx \right)^{1/2} \right)^2 \\ &= \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \|F_k\| \right)^2 = \left(\sum_{k=1}^{\infty} \|F_k\| \right)^2. \end{aligned}$$

This shows that g is integrable function, since $\sum_{k=1}^{\infty} \|F_k\|$ is convergent. Thus, $g(x)$ is finite for any $x \in \mathbb{R}$ and the series $\sum_{k=1}^{\infty} \|F_k(x)\|_{\Omega}$ is convergent for any $x \in \mathbb{R}$. Due to the fact that $\Omega(\mathbb{C})$ is complete, we say that the series $\sum_{k=1}^{\infty} F_k(x)$ is convergent. Let us consider that the function $F : \mathbb{R} \rightarrow \Omega(\mathbb{C})$ defined by

$$F(x) = \begin{cases} \sum_{k=1}^{\infty} F_k(x), & g(x) < \infty \\ \{0\} & , g(x) = \infty \end{cases}$$

From the Proposition 6.1.13 in [12] we say that the set-valued function F is measurable. Since

$$\|F(x)\|_{\Omega}^2 = \left\| \sum_{k=1}^{\infty} F_k(x) \right\|_{\Omega}^2 \leq \left(\sum_{k=1}^{\infty} \|F_k(x)\|_{\Omega} \right)^2 = g(x)$$

and

$$\int_{\mathbb{R}} g(x) dx < \infty$$

we have that $\int_{\mathbb{R}} \|F(x)\|_{\Omega}^2 dx < \infty$. This implies F belongs to $L^2(\mathbb{R}, \Omega(\mathbb{C}))$. Further, for a.e. $x \in \mathbb{R}$

$$\begin{aligned} 0 &\leq \lim_{n \rightarrow \infty} h_{\Omega} \left(\sum_{k=1}^n F_k(x), F(x) \right) \leq \lim_{n \rightarrow \infty} \left\| \sum_{k=1}^n F_k(x) - F(x) \right\|_{\Omega} \\ &= \left\| \lim_{n \rightarrow \infty} \sum_{k=1}^n F_k(x) - \lim_{n \rightarrow \infty} F(x) \right\|_{\Omega} = \left\| \sum_{k=1}^{\infty} F_k(x) - F(x) \right\|_{\Omega} = 0 \end{aligned}$$

and so

$$\lim_{n \rightarrow \infty} h_{\Omega} \left(\sum_{k=1}^n F_k(x), F(x) \right) = 0. \quad (13)$$

Now we shall prove that the function series $\sum_{k=1}^{\infty} F_k$ converges to F in $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ to complete the proof. In accordance with this purpose we will show that $\lim_{n \rightarrow \infty} h_{L^2} \left(\sum_{k=1}^n F_k, F \right) = 0$. Firstly, if we use the Hausdorff metric on $L^2(\mathbb{R}, \Omega(\mathbb{C}))$, we say that for any $\varepsilon > 0$ there exist elements $F_r^i \in L^2(\mathbb{R}, \Omega(\mathbb{C}))$, $i = 1, 2$ such that $\sum_{k=1}^n F_k \preceq F + F_r^1$, $F \preceq \sum_{k=1}^n F_k + F_r^2$ and $\|F_r^i\| \leq r$. Hence $\sum_{k=1}^n F_k(x) \subseteq F(x) + F_r^1(x)$, $F(x) \subseteq \sum_{k=1}^n F_k(x) + F_r^2(x)$ for a.e. $x \in \mathbb{R}$. Further, by the Hausdorff metric on $\Omega(\mathbb{C})$ we have that $\|F_r^i(x)\| \leq h_{\Omega} \left(\sum_{k=1}^n F_k(x), F(x) \right) + r$ for a.e. $x \in \mathbb{R}$ and $i = 1, 2$.

Moreover, for any $r > 0$

$$h_{L^2} \left(\sum_{k=1}^n F_k, F \right) \leq \|F_r^i\| = \left(\int_{\mathbb{R}} \|F_r^i(x)\|_{\Omega}^2 dx \right)^{1/2} \leq \left(\int_{\mathbb{R}} (h_{\Omega} \left(\sum_{k=1}^n F_k(x), F(x) \right) + r)^2 dx \right)^{1/2}.$$

Hence, we have proved the inequality

$$h_{L^2} \left(\sum_{k=1}^n F_k, F \right) \leq \left(\int_{\mathbb{R}} (h_{\Omega} \left(\sum_{k=1}^n F_k(x), F(x) \right))^2 dx \right)^{1/2}.$$

Consequently,

$$\lim_{n \rightarrow \infty} (h_{L^2}(\sum_{k=1}^n F_k, F))^2 \leq \lim_{n \rightarrow \infty} \int_{\mathbb{R}} h_{\Omega}(\sum_{k=1}^n F_k(x), F(x))^2 dx.$$

Using the Theorem 4 the above inequality gives

$$\lim_{n \rightarrow \infty} (h_{L^2}(\sum_{k=1}^n F_k, F))^2 \leq \int_{\mathbb{R}} (\lim_{n \rightarrow \infty} (h_{\Omega}(\sum_{k=1}^n F_k(x), F(x))))^2 dx$$

and this implies $\lim_{n \rightarrow \infty} h_{L^2}(\sum_{k=1}^n F_k, F) = 0$ by the equality (13). So the proof is complete. \square

Theorem 7. *The quasilinear space $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ is a Hilbert quasilinear space with the inner-product given by (4).*

Proof. We know that $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ is a inner-product quasilinear space with respect to the inner-product given by (4). By the definition of norm obtained this inner-product we have that

$$\|F\| = \left\| \left\{ \int_{\mathbb{R}} |f(x)|^2 dx : f \in S^2(F) \right\} \right\|_{\Omega}^{1/2}.$$

Using the Lemma 4 and Theorem 6 show that $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ is complete. Thus, $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ is Hilbert quasilinear space. \square

4 TRANSLATION, MODULATION AND DILATION OPERATORS ON $L^2(\mathbb{R}, \Omega(\mathbb{C}))$

In this section we introduce some important operators on $L^2(\mathbb{R}, \Omega(\mathbb{C}))$.

Definition 16. (Translation) *For $a \in \mathbb{R}$ and $F \in L^2(\mathbb{R}, \Omega(\mathbb{C}))$, the operator \mathcal{T}_a is defined by*

$$(\mathcal{T}_a F)(x) = F(x - a) = \overline{\{(T_a f_n)(x) = f_n(x - a) : f_n \in S^1(F), n = 1, 2, \dots\}} \quad (14)$$

and is called translation by a , where T_a is the translation operator on $L_2(\mathbb{R})$.

Note that $\mathcal{T}_a F$ is defined by the set of translations of countable measurable selections of F . By using the Castaing's theorem (see, [13]) we say that there exists a sequence (f_n) of measurable selections of F such that

$$F(x - a) = \overline{\bigcup_{n \geq 1} (T_a f_n)(x)} = \overline{\bigcup_{n \geq 1} f_n(x - a)}.$$

This implies that $F(x - a)$ which is the translation by $a \in \mathbb{R}$ of a set-valued function $F \in L^2(\mathbb{R}, \Omega(\mathbb{C}))$ can be written as (14). Hence the translation operator \mathcal{T}_a is a natural generalization of classical translation operator T_a in this way.

Notation: We will often write $\mathcal{T}_a F(x)$ instead of $(\mathcal{T}_a F)(x)$ and similarly for the other operators.

Translation operator \mathcal{T}_a is a bounded linear operator between quasilinear spaces: Actually, given any $F, G \in L^2(\mathbb{R}, \Omega(\mathbb{C}))$ and $\lambda \in \mathbb{C}$ we write

$$\mathcal{T}_a(F + G)(x) = (F + G)(x - a) = F(x - a) + G(x - a) = \mathcal{T}_a F(x) + \mathcal{T}_a G(x),$$

$$\mathcal{T}_a(\lambda F)(x) = (\lambda F)(x - a) = \lambda F(x - a) = \lambda \mathcal{T}_a F(x).$$

These show that

$$\mathcal{T}_a(F + G) = \mathcal{T}_a F + \mathcal{T}_a G \text{ and } \mathcal{T}_a(\lambda F) = \lambda \mathcal{T}_a F. \quad (15)$$

Also, if $F \preceq G$ then $F(x) \subseteq G(x)$ for each $x \in \mathbb{R}$. Hence, for any $x \in \mathbb{R}$

$$\mathcal{T}_a F(x) = F(x - a) \subseteq G(x - a) = \mathcal{T}_a G(x). \quad (16)$$

This implies $\mathcal{T}_a F \preceq \mathcal{T}_a G$. By the (15) and (16) we say that \mathcal{T}_a is linear in the meaning of Definition 9. Furthermore, if $F \in L^2(\mathbb{R}, \Omega(\mathbb{C}))$, the change of variable $z = x - a$ shows that

$$\int_{\mathbb{R}} \|\mathcal{T}_a F(x)\|_{\Omega}^2 dx = \int_{\mathbb{R}} \|F(x - a)\|_{\Omega}^2 dx = \int_{\mathbb{R}} \|F(z)\|_{\Omega}^2 dz \quad (17)$$

and so

$$\|\mathcal{T}_a F\| = \|F\|,$$

namely, \mathcal{T}_a is bounded.

Now we will define the modulation and dilation operators in analogy to the definition of translation operator.

Definition 17. (Modulation, Dilation) For a set-valued function $F \in L^2(\mathbb{R}, \Omega(\mathbb{C}))$ we define the following operators:

(i) For $b \in \mathbb{R}$, the operator \mathcal{E}_b is defined by

$$(\mathcal{E}_b F)(x) = e^{2\pi i b x} F(x) = \overline{\{(E_b f_n)(x) = e^{2\pi i b x} f_n(x) : f_n \in S^1(F), n = 1, 2, \dots\}}$$

and is called modulation by b , where E_b is the modulation operator on $L_2(\mathbb{R})$. This definition shows that the modulation operator \mathcal{E}_b is a natural generalization of classical modulation operator E_b .

(ii) For $c \in \mathbb{R}$, the operator \mathcal{D}_c is defined by

$$(\mathcal{D}_c F)(x) = \frac{1}{\sqrt{c}} F\left(\frac{x}{c}\right) = \overline{\{D_c f_n(x) = \frac{1}{\sqrt{c}} f_n\left(\frac{x}{c}\right) : f_n \in S^1(F), n = 1, 2, \dots\}}$$

and is called dilation by c , where D_c is the dilation operator on $L_2(\mathbb{R})$. Thus, we say that the modulation operator \mathcal{D}_c is a natural generalization of classical modulation operator D_c .

It can be easily shown that \mathcal{E}_b and \mathcal{D}_c are bounded linear operators as per above.

Definition 18. Let X_1 and X_2 be Hilbert quasilinear spaces and $T : X_1 \rightarrow X_2$ be a bounded linear operator. The operator $T^* : X_2 \rightarrow X_1$ is called the adjoint operator of T such that for any $x \in X_1$ and $y \in X_2$,

$$\langle Tx, y \rangle = \langle x, T^*y \rangle.$$

Definition 19. Let X be a Hilbert quasilinear space and $T : X \rightarrow X$ a bounded linear operator. The operator T is self-adjoint if $T = T^*$ and is unitary if $TT^* = T^*T = I$.

Proposition 4. *The operators $\mathcal{T}_a, \mathcal{E}_b$ and \mathcal{D}_c are unitary operators from space $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ to $L^2(\mathbb{R}, \Omega(\mathbb{C}))$. Further,*

- $\mathcal{T}_a^{-1} = \mathcal{T}_{-a} = (\mathcal{T}_a)^*$,
- $\mathcal{E}_b^{-1} = \mathcal{E}_{-b} = (\mathcal{E}_b)^*$,
- $\mathcal{D}_c^{-1} = \mathcal{D}_{1/c} = (\mathcal{D}_c)^*$.

Proof. We give a complete proof for the operator \mathcal{T}_a , since the proof is similar to the cases of \mathcal{E}_b and \mathcal{D}_c . Due to the assumption $F \in L^2(\mathbb{R}, \Omega(\mathbb{C}))$ and the equality (17) we say that \mathcal{T}_a maps $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ into $L^2(\mathbb{R}, \Omega(\mathbb{C}))$. Now we will prove that \mathcal{T}_a is unitary: The change of variable $z = x - a$ yields that

$$\begin{aligned} \langle \mathcal{T}_a F, G \rangle &= \int_{\mathbb{R}} \langle \mathcal{T}_a F(x), G(x) \rangle_{\Omega} dx \stackrel{(A)}{=} \int_{\mathbb{R}} \langle F(x - a), G(x) \rangle_{\Omega} dx \\ &\stackrel{(A)}{=} \int_{\mathbb{R}} \langle F(z), G(z + a) \rangle_{\Omega} dz = \langle F, \mathcal{T}_{-a} G \rangle. \end{aligned}$$

Hence by the definition of the adjoint operator \mathcal{T}_a^* we prove that $\mathcal{T}_a^* = \mathcal{T}_{-a}$. Moreover, it is not hard to show that $\mathcal{T}_a \mathcal{T}_a^* = \mathcal{T}_a \mathcal{T}_{-a} = I$ and $\mathcal{T}_a^* \mathcal{T}_a = \mathcal{T}_{-a} \mathcal{T}_a = I$. The calculations show that \mathcal{T}_a is unitary and $\mathcal{T}_a^{-1} = \mathcal{T}_{-a} = (\mathcal{T}_a)^*$. \square

Operators denoted by composition of some of the translation, modulation and dilation operators appear in mathematics and engineering. For this purpose, the following Proposition is useful.

Proposition 5. *For any $a, b \in \mathbb{R}$ and $c > 0$, the following commutation relations hold:*

- (i) $(\mathcal{T}_a \mathcal{E}_b F)(x) = e^{2\pi i b(x-a)} F(x - a) = e^{-2\pi i b a} (\mathcal{E}_b \mathcal{T}_a F)(x)$,
- (ii) $(\mathcal{T}_a \mathcal{D}_c F)(x) = \frac{1}{\sqrt{c}} F\left(\frac{x}{c} - \frac{a}{c}\right) = (\mathcal{D}_c \mathcal{T}_{a/c} F)(x)$,
- (iii) $(\mathcal{D}_{ca} \mathcal{E}_b F)(x) = \frac{1}{\sqrt{c}} e^{2\pi i b/c} F\left(\frac{x}{c}\right) = (\mathcal{E}_{b/c} \mathcal{D}_c F)(x)$.

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У цій статті досліджується важливий простір функцій, який складається з множиннозначних функцій, визначених на множині дійсних чисел зі значеннями у просторі всіх компактних опуклих підмножин комплексних чисел, для яких p -тий степінь їхньої норми інтегрований. Загалом цей простір позначають $L^p(\mathbb{R}, \Omega(\mathbb{C}))$ при $1 \leq p < \infty$ і він має алгебраїчну структуру, його називають квазілінійним простором, що є узагальненням класичного лінійного простору. Далі вводиться скалярний добуток (множиннозначний скалярний добуток) на $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ і, на наш погляд, це важливо для управління інтервальнозначними даними та інтервальною обробкою сигналів. Також це можна використати в терії нечітких сподівань. Визначення скалярного добутку в $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ базується на понятті інтегралу Аумана, який застосовується для інтегрування множиннозначних функцій. Ми показуємо, що простір $L^2(\mathbb{R}, \Omega(\mathbb{C}))$ є гільбертовим квазілінійним простором. Насамкінець ми означаємо оператори перенесення, модуляції та затримки, які є трьома основоположними множиннозначними операторами у гільбертовому квазілінійному просторі $L^2(\mathbb{R}, \Omega(\mathbb{C}))$.

Ключові слова і фрази: гільбертів квазілінійний простір, множиннозначна функція, інтеграл Аумана, перенесення, модуляція, затримка.



MAKHNEI O.V.

MIXED PROBLEM FOR THE SINGULAR PARTIAL DIFFERENTIAL EQUATION OF PARABOLIC TYPE

The scheme for solving of a mixed problem is proposed for a differential equation

$$a(x) \frac{\partial T}{\partial \tau} = \frac{\partial}{\partial x} \left(c(x) \frac{\partial T}{\partial x} \right) - g(x) T$$

with coefficients $a(x)$, $g(x)$ that are the generalized derivatives of functions of bounded variation, $c(x) > 0$, $c^{-1}(x)$ is a bounded and measurable function. The boundary and initial conditions have the form

$$\begin{cases} p_1 T(0, \tau) + p_2 T_x^{[1]}(0, \tau) = \psi_1(\tau), \\ q_1 T(l, \tau) + q_2 T_x^{[1]}(l, \tau) = \psi_2(\tau), \\ T(x, 0) = \varphi(x), \end{cases}$$

where $p_1 p_2 \leq 0$, $q_1 q_2 \geq 0$ and by $T_x^{[1]}(x, \tau)$ we denote the quasiderivative $c(x) \frac{\partial T}{\partial x}$. A solution of this problem we seek by the reduction method in the form of sum of two functions $T(x, \tau) = u(x, \tau) + v(x, \tau)$. This method allows to reduce solving of proposed problem to solving of two problems: a quasistationary boundary problem with initial and boundary conditions for the search of the function $u(x, \tau)$ and a mixed problem with zero boundary conditions for some inhomogeneous equation with an unknown function $v(x, \tau)$. The first of these problems is solved through the introduction of the quasiderivative. Fourier method and expansions in eigenfunctions of some boundary value problem for the second-order quasidifferential equation $(c(x)X'(x))' - g(x)X(x) + \omega a(x)X(x) = 0$ are used for solving of the second problem. The function $v(x, \tau)$ is represented as a series in eigenfunctions of this boundary value problem. The results can be used in the investigation process of heat transfer in a multilayer plate.

Key words and phrases: mixed problem, quasiderivative, eigenfunctions, Fourier method.

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INTRODUCTION

Boundary problems for differential equations of heat conduction with smooth coefficients were studied quite comprehensively in the literature (e.g., see [7]). However, during the modeling of heat transfer processes, the boundary problems with piecewise continuous coefficients or coefficients that have generalized derivatives of discontinuous functions are often appeared. Such problems have already begun to be studied in the works [2, 5, 6].

The present paper deals with solving of a mixed problem for a partial differential equation of parabolic type with coefficients that are the generalized derivatives of functions of bounded variation. A reduction method [7] is used for solving of this problem. This method allows to

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reduce solving of this problem to solving of two problems: a quasistationary boundary problem with initial and boundary conditions and a mixed problem with zero boundary conditions for some inhomogeneous equation. Fourier method and expansions in eigenfunctions of some boundary value problem for the second-order quasidifferential equation are used for solving of the second of these problems. In this paper we consider a more general statement of the problem than in [2]. Moreover, it is proved the non-negativity of the eigenvalues, which is a necessary condition for the correctness of the description of the heat transfer process.

Quasidifferential equations are equations that contain terms of the form $(p(x)y^{(m)})^{(n)}$. These equations cannot be reduced to conventional differential equations by n -fold differentiation if the coefficient $p(x)$ is not sufficiently smooth. The introduction of quasiderivatives is used for their research [3].

1 FORMULATION OF THE PROBLEM

Consider the next mixed problem for a partial differential equation of parabolic type. It is necessary to find a solution $T(x, \tau)$ of the equation

$$a(x) \frac{\partial T}{\partial \tau} = \frac{\partial}{\partial x} \left(c(x) \frac{\partial T}{\partial x} \right) - g(x) T \quad (1)$$

with boundary conditions

$$\begin{cases} p_1 T(0, \tau) + p_2 T_x^{[1]}(0, \tau) = \psi_1(\tau), \\ q_1 T(l, \tau) + q_2 T_x^{[1]}(l, \tau) = \psi_2(\tau) \end{cases} \quad (2)$$

and initial condition

$$T(x, 0) = \varphi(x), \quad (3)$$

where $a(x) = b'(x)$, $g(x) = h'(x)$, $b(x)$, $h(x)$ are right continuous nondecreasing real functions of bounded variation on the interval $[0, l]$, $c(x) > 0$, $c^{-1}(x)$ is a bounded and measurable function on the interval $[0, l]$, $\varphi(x)$ is a continuous function on the interval $[0, l]$, $\psi_1(\tau)$ and $\psi_2(\tau)$ are continuously differentiable functions for $\tau \geq 0$, p_1, p_2, q_1, q_2 are real numbers, $p_1 p_2 \leq 0$, $q_1 q_2 \geq 0$. By $T_x^{[1]}(x, \tau) \stackrel{df}{=} c(x) \frac{\partial T}{\partial x}$ we denote the quasiderivative. The primes in the formulas $a(x) = b'(x)$, $g(x) = h'(x)$ stand for the generalized differentiation, and hence the functions $a(x)$, $g(x)$ are measures, i.e., a zero-order distributions on the space of continuous compactly supported functions [1].

A solution of problem (1)–(3) seek by the reduction method in the form of sum of two functions

$$T(x, \tau) = u(x, \tau) + v(x, \tau). \quad (4)$$

Any of functions u or v can be chosen by a special way, then another one will be determined uniquely.

2 QUASISTATIONARY BOUNDARY PROBLEM FOR $u(x, \tau)$

We define $u(x, \tau)$ as the solution of the boundary problem

$$\frac{\partial}{\partial x} \left(c(x) \frac{\partial u}{\partial x} \right) - g(x) u = 0, \quad (5)$$

$$\begin{cases} p_1 u(0, \tau) + p_2 u_x^{[1]}(0, \tau) = \psi_1(\tau), \\ q_1 u(l, \tau) + q_2 u_x^{[1]}(l, \tau) = \psi_2(\tau), \end{cases} \quad (6)$$

which is derived from problem (1)–(3), if τ is a parameter. Here the quasiderivative $u_x^{[1]}(x, \tau) \stackrel{df}{=} c(x) \frac{\partial u}{\partial x}$, then $\frac{\partial u}{\partial x} = \frac{u^{[1]}}{c(x)}$. With the help of the vector $\bar{u} = (u, u^{[1]})^T$ equation (5) is reduced to the system

$$\begin{pmatrix} u \\ u^{[1]} \end{pmatrix}' = \begin{pmatrix} 0 & \frac{1}{c(x)} \\ g(x) & 0 \end{pmatrix} \begin{pmatrix} u \\ u^{[1]} \end{pmatrix}. \quad (7)$$

Boundary conditions (6) are also represented in the vector form

$$P \cdot \bar{u}(0, \tau) + Q \cdot \bar{u}(l, \tau) = \bar{\Gamma}(\tau), \quad (8)$$

where

$$P = \begin{pmatrix} p_1 & p_2 \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 \\ q_1 & q_2 \end{pmatrix}, \quad \bar{\Gamma}(\tau) = \begin{pmatrix} \psi_1(\tau) \\ \psi_2(\tau) \end{pmatrix}, \quad \bar{u} = \begin{pmatrix} u \\ u^{[1]} \end{pmatrix}.$$

Suppose that we know the Cauchy matrix $B(x, s)$ of system (7). It can be constructed, for example, when the coefficients $c(x)$ and $g(x)$ are piecewise constant functions [4]. In the general case, it is necessary to know the fundamental system of solutions of system (7) for construction the Cauchy matrix [4]. By [4], equation (5) with arbitrary initial conditions $u(x_0, \tau) = u_0$, $u^{[1]}(x_0, \tau) = u_1$, $x_0 \in [0, l]$ has a unique solution in the class of absolutely continuous functions, and the quasiderivative $u^{[1]}$ of this solution has a bounded variation by the variable x on the interval $[0, l]$.

We have $\bar{u}(x, \tau) = B(x, 0)\bar{u}_0$, where $\bar{u}_0 = \bar{u}(0, \tau)$. We shall determine \bar{u}_0 . From boundary conditions (8) we obtain $P \cdot \bar{u}_0 + Q \cdot B(l, 0) \cdot \bar{u}_0 = \bar{\Gamma}$ whence $\bar{u}_0 = (P + Q \cdot B(l, 0))^{-1} \cdot \bar{\Gamma}$. Therefore,

$$\bar{u}(x, \tau) = B(x, 0) \cdot (P + Q \cdot B(l, 0))^{-1} \cdot \bar{\Gamma}(\tau). \quad (9)$$

3 MIXED PROBLEM FOR $v(x, \tau)$

We substitute $u(x, \tau)$ and $v(x, \tau)$ into equation (1)

$$a(x) \left(\frac{\partial u}{\partial \tau} + \frac{\partial v}{\partial \tau} \right) = \frac{\partial}{\partial x} \left(c(x) \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \right) \right) - g(x)(u + v).$$

In consequence of (5) we have the equation

$$a(x) \frac{\partial v}{\partial \tau} = \frac{\partial}{\partial x} \left(c(x) \frac{\partial v}{\partial x} \right) - g(x)v - a(x) \frac{\partial u}{\partial \tau}. \quad (10)$$

According to formula (9) the derivative $\frac{\partial u}{\partial \tau}$ is a continuous function of the variable x on $[0, l]$ and so the last term in equation (10) is correct.

By taking into account formula (4), we define the boundary conditions for v from conditions (2)

$$\begin{aligned} p_1 u(0, \tau) + p_2 u_x^{[1]}(0, \tau) + p_1 v(0, \tau) + p_2 v_x^{[1]}(0, \tau) &= \psi_1(\tau), \\ q_1 u(l, \tau) + q_2 u_x^{[1]}(l, \tau) + q_1 v(l, \tau) + q_2 v_x^{[1]}(l, \tau) &= \psi_2(\tau). \end{aligned}$$

By virtue of (6), we obtain

$$\begin{cases} p_1 v(0, \tau) + p_2 v_x^{[1]}(0, \tau) = 0, \\ q_1 v(l, \tau) + q_2 v_x^{[1]}(l, \tau) = 0. \end{cases} \quad (11)$$

The initial condition is determined similarly

$$v(x, 0) = \varphi(x) - u(x, 0) \stackrel{df}{=} \tilde{\varphi}(x). \quad (12)$$

4 FOURIER METHOD AND EIGENVALUE PROBLEM

We search for non-trivial solutions of the homogeneous differential equation

$$a(x) \frac{\partial v}{\partial \tau} = \frac{\partial}{\partial x} \left(c(x) \frac{\partial v}{\partial x} \right) - g(x)v$$

with boundary conditions (11) in the form

$$v(x, \tau) = e^{-\omega \tau} X(x), \quad (13)$$

where ω is a parameter, and $X(x)$ is a function. Then

$$-\omega a(x) e^{-\omega \tau} X(x) = (c(x) X'(x))' e^{-\omega \tau} - g(x) X(x) e^{-\omega \tau},$$

whence we get the quasidifferential equation

$$(c(x) X'(x))' - g(x) X(x) + \omega a(x) X(x) = 0. \quad (14)$$

Substituting formula (13) in boundary conditions (11), we obtain

$$\begin{cases} p_1 X(0) + p_2 X^{[1]}(0) = 0, \\ q_1 X(l) + q_2 X^{[1]}(l) = 0. \end{cases} \quad (15)$$

We denote by ω_k the eigenvalues of boundary problem (14), (15). Let $X_k(\omega_k, x)$ be the corresponding eigenfunctions, $k = 1, 2, \dots, \infty$.

By [8], all eigenvalues ω_k of boundary problem (14), (15) are real, there are a countable number of them, and their set has not a finite limit point. The eigenfunctions $X_k(\omega_k, x)$ that are corresponded to the different eigenvalues are orthogonal in the sense

$$\int_0^l X_m(\omega_m, x) X_n(\omega_n, x) db(x) = 0, \quad \omega_m \neq \omega_n.$$

We now prove that all of the eigenvalues ω_k of boundary value problem (14), (15) are non-negative with the coefficients imposed in section 1.

To do this, we multiply both parts of the equation

$$(c(x) X_k'(x))' - g(x) X_k(x) + \omega_k a(x) X_k(x) = 0$$

by $X_k(x)$

$$(c(x) X_k'(x))' X_k(x) - g(x) X_k^2(x) + \omega_k a(x) X_k^2(x) = 0.$$

Then, taking into account that $X_k^{[1]}(x) = c(x)X_k'(x)$, after transformations we obtain

$$\omega_k a(x)X_k^2(x) = -(X_k^{[1]}(x)X_k(x))' + X_k^{[1]}(x)X_k'(x) + g(x)X_k^2(x).$$

By integrating both parts of the obtained ratio in the range from 0 to l , we will have

$$\begin{aligned} \omega_k \int_0^l X_k^2(x)db(x) &= -X_k^{[1]}(l)X_k(l) + X_k^{[1]}(0)X_k(0) \\ &+ \int_0^l c(x)(X_k'(x))^2 dx + \int_0^l X_k^2(x)dh(x). \end{aligned} \tag{16}$$

Since the functions $X_k(x)$ are absolutely continuous as a result of [4], and their quasiderivatives $X_k^{[1]}(x)$ have a bounded variation on the interval $[0, l]$, we see that all of the above transformations have sense.

All integrals in formula (16) are non-negative. If $p_1 = 0$ or $p_2 = 0$, then $X_k^{[1]}(0)X_k(0) = 0$. If $p_1 p_2 < 0$, then from the first condition of system (15) we have $X_k^{[1]}(0) = -\frac{p_1}{p_2}X_k(0)$. Then $X_k^{[1]}(0)X_k(0) = -\frac{p_1}{p_2}X_k^2(0) \geq 0$. Similarly, if $q_1 = 0$ or $q_2 = 0$, then $X_k^{[1]}(l)X_k(l) = 0$, otherwise $X_k^{[1]}(l)X_k(l) = -\frac{q_1}{q_2}X_k^2(l) \leq 0$. Consequently, it follows from formula (16) that all $\omega_k \geq 0$.

5 METHOD OF THE EIGENFUNCTIONS

We seek $v(x, \tau)$ in the form of the series

$$v(x, \tau) = \sum_{k=1}^{\infty} t_k(\tau)X_k(\omega_k, x), \tag{17}$$

where $X_k(\omega_k, x)$ are the eigenfunctions of boundary problem (14), (15). We substitute formula (17) into equation (10)

$$\begin{aligned} a(x) \frac{\partial}{\partial \tau} \left(\sum_{k=1}^{\infty} t_k(\tau)X_k \right) &= \frac{\partial}{\partial x} \left(c(x) \frac{\partial}{\partial x} \left(\sum_{k=1}^{\infty} t_k(\tau)X_k \right) \right) \\ &- g(x) \sum_{k=1}^{\infty} t_k(\tau)X_k - a(x) \frac{\partial u}{\partial \tau}, \end{aligned}$$

whence, under the assumption of uniform convergence of series (17) and series derived from it by differentiation by x or τ , we have

$$a(x) \sum_{k=1}^{\infty} t_k'(\tau)X_k = \sum_{k=1}^{\infty} t_k(\tau) \left((c(x)X_k')' - g(x)X_k \right) - a(x) \frac{\partial u}{\partial \tau}.$$

As a result of equation (14) there is equality

$$(c(x)X_k')' - g(x)X_k = -\omega_k a(x)X_k,$$

then

$$a(x) \sum_{k=1}^{\infty} t_k'(\tau)X_k = - \sum_{k=1}^{\infty} t_k(\tau) \omega_k a(x)X_k - a(x) \frac{\partial u}{\partial \tau}.$$

Therefore,

$$\sum_{k=1}^{\infty} (t'_k(\tau) + \omega_k t_k(\tau)) X_k = -\frac{\partial u}{\partial \tau}. \quad (18)$$

We expand the known function $\frac{\partial u}{\partial \tau}$ in a series in the eigenfunctions of boundary problem (14), (15):

$$\frac{\partial u}{\partial \tau} = \sum_{k=1}^{\infty} d_k(\tau) X_k(\omega_k, x), \quad (19)$$

where

$$d_k(\tau) = \frac{1}{\|X_k\|} \int_0^l \frac{\partial u}{\partial \tau} X_k(\omega_k, x) db(x), \quad \|X_k\| = \int_0^l X_k^2(\omega_k, x) db(x).$$

By substituting formula (19) into (18), we obtain

$$t'_k(\tau) + \omega_k t_k(\tau) = -d_k(\tau), \quad k = 1, 2, \dots, \infty. \quad (20)$$

Since formulas (12) and (17), we have

$$v(x, 0) = \sum_{k=1}^{\infty} t_k(0) X_k(\omega_k, x) \equiv \tilde{\varphi}(x).$$

We expand the function $\tilde{\varphi}(x)$ in a series in the eigenfunctions

$$\tilde{\varphi}(x) = \sum_{k=1}^{\infty} \varphi_k X_k(\omega_k, x), \quad \varphi_k = \frac{1}{\|X_k\|} \int_0^l \tilde{\varphi}(x) X_k(\omega_k, x) db(x).$$

Consequently,

$$t_k(0) = \varphi_k, \quad k = 1, 2, \dots, \infty. \quad (21)$$

Then for all positive integer k we have Cauchy problems (20), (21) for ordinary differential equations.

General solutions of linear inhomogeneous equations (20) acquire the formulas

$$t_k(\tau) = \left(C_k - \int_0^{\tau} d_k(s) e^{\omega_k s} ds \right) e^{-\omega_k \tau},$$

where C_k are arbitrary constants. Therefore, by using initial conditions (21), we find for each positive integer k the solution of the corresponding Cauchy problem

$$t_k(\tau) = \varphi_k e^{-\omega_k \tau} - \int_0^{\tau} d_k(s) e^{\omega_k(s-\tau)} ds.$$

Then, by virtue of formula (17), we obtain

$$v(x, \tau) = \sum_{k=1}^{\infty} \left(\varphi_k e^{-\omega_k \tau} - \int_0^{\tau} d_k(s) e^{\omega_k(s-\tau)} ds \right) X_k(\omega_k, x).$$

Thus, by using the reduction method, Fourier method and the expansion in a series in eigenfunctions, we built the solution of the boundary problem for the parabolic type partial differential equation with distributions. The results can be used in the investigation of the process of heat transfer in a multilayer plate.

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Запропоновано схему розв'язування мішаної задачі для диференціального рівняння

$$a(x) \frac{\partial T}{\partial \tau} = \frac{\partial}{\partial x} \left(c(x) \frac{\partial T}{\partial x} \right) - g(x) T$$

з коефіцієнтами $a(x)$, $g(x)$, які є узагальненими похідними функцій обмеженої варіації, $c(x) > 0$, $c^{-1}(x)$ — обмежена і вимірна функція. Крайові і початкова умови мають вигляд

$$\begin{cases} p_1 T(0, \tau) + p_2 T_x^{[1]}(0, \tau) = \psi_1(\tau), \\ q_1 T(l, \tau) + q_2 T_x^{[1]}(l, \tau) = \psi_2(\tau), \end{cases}$$

$$T(x, 0) = \varphi(x),$$

де $p_1 p_2 \leq 0$, $q_1 q_2 \geq 0$, а через $T_x^{[1]}(x, \tau)$ позначено квазіпохідну $c(x) \frac{\partial T}{\partial x}$. Розв'язок цієї задачі шукається методом редукції у вигляді суми двох функцій $T(x, \tau) = u(x, \tau) + v(x, \tau)$. Цей метод дає змогу звести розв'язування поставленої задачі до розв'язування двох задач: крайової квазістаціонарної задачі з початковими і крайовими умовами для відшукування функції $u(x, \tau)$ і мішаної задачі з нульовими крайовими умовами для деякого неоднорідного рівняння з невідомою функцією $v(x, \tau)$. Перша з цих задач розв'язується з допомогою введення квазіпохідної. Для розв'язування другої задачі застосовується метод Фур'є і розвинення за власними функціями деякої крайової задачі для квазидиференціального рівняння другого порядку $(c(x)X'(x))' - g(x)X(x) + \omega a(x)X(x) = 0$. Функція $v(x, \tau)$ подається у вигляді ряду за власними функціями цієї крайової задачі. Отримані результати можна використовувати для дослідження процесу теплопередачі в багатошаровій плиті.

Ключові слова і фрази: мішана задача, квазіпохідна, власні функції, метод Фур'є.

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INVARIANT IDEMPOTENT MEASURES

The idempotent mathematics is a part of mathematics in which arithmetic operations in the reals are replaced by idempotent operations. In the idempotent mathematics, the notion of idempotent measure (Maslov measure) is a counterpart of the notion of probability measure. The idempotent measures found numerous applications in mathematics and related areas, in particular, the optimization theory, mathematical morphology, and game theory.

In this note we introduce the notion of invariant idempotent measure for an iterated function system in a complete metric space. This is an idempotent counterpart of the notion of invariant probability measure defined by Hutchinson. Remark that the notion of invariant idempotent measure was previously considered by the authors for the class of ultrametric spaces.

One of the main results is the existence and uniqueness theorem for the invariant idempotent measures in complete metric spaces. Unlikely to the corresponding Hutchinson's result for invariant probability measures, our proof does not rely on metrization of the space of idempotent measures.

An analogous result can be also proved for the so-called in-homogeneous idempotent measures in complete metric spaces.

Also, our considerations can be extended to the case of the max-min measures in complete metric spaces.

Key words and phrases: idempotent measure (Maslov measure), iterated function system, invariant measure.

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INTRODUCTION

The idempotent mathematics is a part of mathematics in which arithmetic operations on the reals are replaced by idempotent operations (e.g., max, min; see [9]). According to an informal correspondence principle, every substantial notion of the (ordinary) mathematics has its counterpart in the idempotent mathematics. In this way we obtain the notion of idempotent measure, which is an idempotent analogue of that of probability measure. The idempotent measures found numerous applications, e.g. in the optimization theory, mathematical morphology, and game theory (see [2, 12–15]).

Different aspects of the theory of idempotent measures are considered in [1, 5, 8, 22]. In particular, the topology of spaces of the idempotent measures on some compact metric spaces is investigated in [5]. However, the theory of idempotent measures is considerably less developed than that of probability measures.

The mathematical foundations of the theory of deterministic fractals were created by Hutchinson [16]. In particular, he introduced the notions of invariant (self-similar) set and invariant measure for an iterated function system (IFS) of contractions on a complete metric space.

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The existence of invariant measures is proved in [16] by using the Banach contraction principle for suitable metrization of the set of probability measures on a metric space. The invariant measures impose an additional structure on the invariant set for the given IFS.

In [4], the authors considered a modification of the notions of invariant set and invariant probability measure, namely, the notions of in-homogeneous set and in-homogeneous probability measure (see also [17,18]). The inhomogeneous sets and measures are used, in particular, in image compression (see, e.g., [19]).

The aim of this note is to introduce the invariant idempotent measures for given IFS. In the case of idempotent measure, we use the weak* convergence for proving the existence of invariant element. This approach seems to be fairly general and we anticipate new results in this direction (see the concluding remarks).

Note also that the invariant idempotent measures on ultrametric spaces are introduced and investigated in [11].

1 PRELIMINARIES

As usual, $C(X)$ denotes the Banach space of continuous functions on a compact space X . We endow $C(X)$ with the sup-norm. For any $c \in \mathbb{R}$, by c_X we denote the constant function on X taking the value c .

By \bar{A} we denote the closure of a set A in a topological space.

Let $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$. We use the following operations \odot, \oplus of idempotent mathematics (see e.g., [9]): $x \odot y = x + y$, and $x \oplus y = \max\{x, y\}$, $x, y \in \mathbb{R}_{\max}$ (convention: $(-\infty) \odot x = x \odot (-\infty) = -\infty$, $(-\infty) \oplus x = x \oplus (-\infty) = x$). Also we consider the operations $\odot: \mathbb{R} \times C(X) \rightarrow C(X)$, $\lambda \odot \varphi = \lambda_X + \varphi$, and $\oplus: C(X) \times C(X) \rightarrow C(X)$, $(\varphi \oplus \psi) = \max\{\varphi, \psi\}$.

Definition 1.1. A functional $\mu: C(X) \rightarrow \mathbb{R}$ is called an idempotent measure (a Maslov measure) if

1. $\mu(c_X) = c$,
2. $\mu(c \odot \varphi) = c \odot \mu(\varphi)$, and
3. $\mu(\varphi \oplus \psi) = \mu(\varphi) \oplus \mu(\psi)$

(see, e.g., [22] and references therein for the history and motivations of the notion of Maslov measure and Maslov integral).

By $I(X)$ we denote the set of all idempotent measures on X .

Let δ_x (or $\delta(x)$) denote the Dirac measure concentrated at $x \in X$, i.e., $\delta_x(\varphi) = \varphi(x)$, $\varphi \in C(X)$. Clearly, $\delta_x \in I(X)$. A more complicated example of an idempotent measure is $\mu = \bigoplus_{i=1}^n \alpha_i \odot \delta_{x_i}$, where $x_i \in X$ and $\alpha_i \in \mathbb{R}_{\max}$, $i = 1, \dots, n$, and $\bigoplus_{i=1}^n \alpha_i = 0$.

We endow the set $I(X)$ with the weak* topology. In the case of compact metrizable space X , this topology is completely described by the convergent sequences: $(\mu_i)_{i=1}^{\infty}$ converges to μ if and only if $\lim_{i \rightarrow \infty} \mu_i(\varphi) = \mu(\varphi)$, for all $\varphi \in C(X)$.

Given a map $f: X \rightarrow Y$ of compact Hausdorff spaces, the map $I(f): I(X) \rightarrow I(Y)$ is defined by the formula $I(f)(\mu)(\varphi) = \mu(\varphi f)$, for every $\mu \in I(X)$ and $\varphi \in C(Y)$. That $I(f)$ is continuous and that I is a covariant functor acting in the category **Comp** of compact Hausdorff spaces and continuous maps was proved in [22].

If $f: A \rightarrow X$ is an embedding of compact Hausdorff spaces, then so is the map $I(f): I(A) \rightarrow I(X)$. We identify $I(A)$ and the subspace $I(f)(I(A))$ via this embedding. The *support* $\text{supp}(\mu)$ of an idempotent measure $\mu \in I(X)$ is the minimal (with respect to inclusion) closed subset A in X such that $\mu \in I(A)$. According to [7] one can define the space $I(X)$ also in non-compact case. If X is a Tychonov space, then let

$$I(X) = \{\mu \in I(\beta X) \mid \text{supp}(\mu) \subset X \subset \beta X\},$$

where βX stands for the Stone-Ćech compactification of X .

Recall that a map $f: X \rightarrow Y$ of a metric space (X, d) into a metric space (Y, ρ) is called a contraction if there exists $c \in (0, 1)$ such that $\rho(f(x), f(y)) \leq cd(x, y)$, for all $x, y \in X$.

By $\exp X$ we denote the hyperspace of a topological space X , i.e., the set of all nonempty compact subsets of X . If (X, d) is a metric space, then $\exp X$ is endowed with the Hausdorff metric d_H ,

$$d_H(A, B) = \inf\{\varepsilon > 0 \mid A \subset O_\varepsilon(B), B \subset O_\varepsilon(A)\},$$

where $O_r(C)$ stands for the r -neighborhood of a set C in X .

2 RESULT

Let X be a complete metric space and let f_1, \dots, f_n be an Iterated Function System (thereafter IFS) on X . We assume that all f_i are contractions. Let also $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ be such that $\bigoplus_{i=1}^n \alpha_i = 0$.

We denote by Ψ_0 the identity map of $\exp X$ and, for $i > 0$, define $\Psi_i: \exp X \rightarrow \exp X$ inductively: $\Psi_i(A) = \bigcup_{j=1}^n f_j(\Psi_{i-1}(A))$.

Let $\Phi_0: I(X) \rightarrow I(X)$ be the identity map. For $i > 0$, define $\Phi_i: I(X) \rightarrow I(X)$ inductively: $\Phi_i(\mu) = \bigoplus_{j=1}^n \alpha_j \odot I(f_j)(\Phi_{i-1}(\mu))$. Thus, $\Phi_i = \Phi_1 \Phi_1 \cdots \Phi_1$ (i times). It is easy to check that the maps Φ_i are well-defined. In this case, we say that $\mu \in I(X)$ is an *invariant idempotent measure* if $\Phi_i(\mu) = \mu$ for every $i = 0, 1, \dots$ (equivalently, $\Phi_1(\mu) = \mu$).

Now, let $\tau \in I(X)$ and let $\alpha_1, \dots, \alpha_n, \alpha \in \mathbb{R}$ be such that $(\bigoplus_{i=1}^n \alpha_i) \oplus \alpha = 0$. Let $\hat{\Phi}_0 = \Phi_0$ and define $\hat{\Phi}_i: I(X) \rightarrow I(X)$ inductively: $\hat{\Phi}_i(\mu) = \bigoplus_{j=1}^n \alpha_j \odot I(f_j)(\hat{\Phi}_{i-1}(\mu)) \oplus \alpha \odot \tau$. Following the terminology of [17, 18] we say that $\hat{\mu} \in I(X)$ is an *inhomogeneous invariant idempotent measure* if $\hat{\mu} = \hat{\Phi}_1(\hat{\mu})$.

Theorem 1. *There exists a unique invariant idempotent measure for the IFS f_1, \dots, f_n and $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ with $\bigoplus_{i=1}^n \alpha_i = 0$. This invariant measure is the limit of the sequence $(\Phi_i(\mu))_{i=1}^\infty$, for arbitrary $\mu \in I(X)$.*

Proof. Let $\mu \in I(X)$. We are going to prove that the sequence $(\Phi_i(\mu)(\varphi))_{i=1}^\infty$ converges for arbitrary $\varphi \in C(X)$.

We first note that, without loss of generality, one may assume that X is compact. Indeed, for every $i \geq 0$, we see that

$$\text{supp}(\Phi_i(\mu)) \subset \Psi_i(\text{supp}(\mu)) \subset \overline{\bigcup_{j=0}^{\infty} \Psi_j(\text{supp}(\mu))}$$

and the latter set is compact by [16].

Let $\varphi \in C(X)$ and let $\varepsilon > 0$. There exists $\eta > 0$ such that, for every $A \subset X$, $\text{diam}(A) < \eta$ implies $\text{diam}(\varphi(A)) < \varepsilon$. There exists $N \in \mathbb{N}$ such that for every $k \geq N$,

$$\text{diam}(f_{i_1} \dots f_{i_k}(X)) < \eta,$$

for every $i_1, \dots, i_k \in \{1, \dots, n\}$. Then

$$\begin{aligned} \Phi_N(\mu)(\varphi) &= \bigoplus_{i_1, \dots, i_N} (\alpha_{i_1} \odot \dots \odot \alpha_{i_N}) \odot \mu(\varphi f_{i_1} \dots f_{i_N}) \\ &= (\alpha_{j_1} \odot \dots \odot \alpha_{j_N}) \odot \mu(\varphi f_{j_1} \dots f_{j_N}) \end{aligned}$$

for some j_1, \dots, j_N . By the choice of N ,

$$\varphi(x) - \varepsilon < \mu(\varphi f_{i_1} \dots f_{i_N}) < \varphi(x) + \varepsilon, \tag{1}$$

for every $x \in f_{j_1} \dots f_{j_N}(X)$. There is j such that $\alpha_j = 0$. Then, for every $k \geq 1$,

$$\Phi_{N+k}(\mu)(\varphi) \geq (\alpha_{j_1} \odot \dots \odot \alpha_{j_N}) \odot \mu(\varphi f_{j_1} \dots f_{j_N} \underbrace{f_j \dots f_j}_k).$$

Then also $\varphi(x) - \varepsilon < \mu(\varphi f_{j_1} \dots f_{j_N} \underbrace{f_j \dots f_j}_k) < \varphi(x) + \varepsilon$, for every $x \in f_{j_1} \dots f_{j_N} \underbrace{f_j \dots f_j}_k(X) \subset f_{j_1} \dots f_{j_N}(X)$. We conclude that $\Phi_{N+k}(\mu)(\varphi) \geq \Phi_N(\mu)(\varphi) - 2\varepsilon$ and, since the sequence $(\Phi_i(\mu))$ is bounded, we conclude that there exists the limit of this sequence.

Now we are going to prove that the limit does not depend on the choice of μ . Let also $\nu \in I(X)$. Again, without loss of generality, one may assume that X is compact. Indeed, one could let

$$X = \overline{\bigcup_{j=0}^{\infty} \Psi_j(\text{supp}(\mu) \cup \text{supp}(\nu))}.$$

Replacing μ by ν in (1) we obtain $\Phi_{N+k}(\mu)(\varphi) \geq \Phi_N(\nu)(\varphi) - 2\varepsilon$ and therefore $\lim_{k \rightarrow \infty} \Phi_k(\mu)(\varphi) = \lim_{k \rightarrow \infty} \Phi_{N+k}(\mu) \geq \Phi_N(\nu)(\varphi) - 2\varepsilon$. From the latter inequality we obtain

$$\lim_{k \rightarrow \infty} \Phi_k(\mu)(\varphi) \geq \lim_{N \rightarrow \infty} \Phi_N(\nu)(\varphi) - 2\varepsilon$$

and, because of arbitrariness of $\varepsilon > 0$, $\lim_{k \rightarrow \infty} \Phi_k(\mu)(\varphi) \geq \lim_{N \rightarrow \infty} \Phi_N(\nu)(\varphi)$.

Switching μ and ν we obtain the reverse inequality and therefore the equality.

Finally, the uniqueness of the invariant idempotent measure is an obvious consequence of the above established fact that the limit $\lim_{i \rightarrow \infty} \Phi_i(\mu)$ does not depend on the choice of μ . □

Example 1. Let $X = [0, 1]$ and let $f_1, f_2: X \rightarrow X$ be given by the formulas: $f_1(t) = t/3$, $f_2(t) = (t + 2)/3$. The invariant set that corresponds to the IFS f_1, f_2 is exactly the middle-third Cantor set.

Let $\alpha_1 = 0$ and $\alpha_2 = -1$. Let $\mu = \delta_0$. Then, for every $n \geq 1$,

$$\mu_n = 0 \odot \delta_0 \oplus \bigoplus_{1 \leq i_1 < \dots < i_k \leq n} (-k) \odot \delta \left(\sum_{j=1}^k \frac{2}{3^{i_j}} \right).$$

Then the invariant idempotent measure corresponding to $\{f_1, f_2; \alpha_1, \alpha_2\}$ is

$$\lim_{n \rightarrow \infty} \mu_n = 0 \odot \delta_0 \oplus \bigoplus_{1 \leq i_1 < \dots < i_k} (-k) \odot \delta \left(\sum_{j=1}^k \frac{2}{3^{i_j}} \right).$$

One can similarly prove the following result.

Theorem 2. *There exists a unique inhomogeneous invariant idempotent measure for the IFS f_1, \dots, f_n and $\alpha_1, \dots, \alpha_n, \alpha \in \mathbb{R}$ with $(\bigoplus_{i=1}^n \alpha_i) \oplus \alpha = 0$. This inhomogeneous invariant measure is the limit of the sequence $(\hat{\Phi}_i(\mu))_{i=1}^\infty$, for arbitrary $\mu \in I(X)$.*

3 MAX-MIN MEASURES

Let $\bar{\mathbb{R}} = \mathbb{R}_{\max} \cup \{\infty\} = \mathbb{R} \cup \{-\infty, \infty\}$. In the sequel, \otimes is used for min.

A functional $\mu: C(X) \rightarrow \mathbb{R}$ is called a *max-min measure* if the following are satisfied:

1. $\mu(c_x) = c$;
2. $\mu(\varphi \oplus \psi) = \mu(\varphi) \oplus \mu(\psi)$;
3. $\mu(c \otimes \varphi) = c \otimes \mu(\varphi)$

(see, e.g., [6] for details).

By $J(X)$ we denote the set of all max-min measures on a compact Hausdorff space X . The set $J(X)$ is endowed with the weak*-topology. A base of this topology consists of the sets of the form

$$\{\mu \in J(X) \mid |\mu(\varphi_i) - \nu(\varphi_i)| < \varepsilon, i = 1, \dots, n\},$$

where $\nu \in J(X)$, $\varphi_i \in C(X)$, $i = 1, \dots, n$, $n \in \mathbb{N}$. Every map $f: X \rightarrow Y$ of compact Hausdorff spaces induces a map $J(f): J(X) \rightarrow J(Y)$ defined as follows: $J(f)(\mu)(\varphi) = \mu(\varphi f)$. It is proved in [6] that J is a functor acting in the category **Comp**. Similarly as above, one can consider the spaces $J(X)$ for Tychonov (in particular, metrizable) spaces X .

Let X be a complete metric space and let f_1, \dots, f_n be an IFS on X . We assume that all f_i are contractions. Let also $\alpha_1, \dots, \alpha_n \in \bar{\mathbb{R}}$ be such that $\bigoplus_{i=1}^n \alpha_i = \infty$.

Let $\Phi'_0: J(X) \rightarrow J(X)$ be the identity map. For $i > 0$, define $\Phi'_i: J(X) \rightarrow J(X)$ inductively: $\Phi'_i(\mu) = \bigoplus_{j=1}^n \alpha_j \otimes J(f_j)(\Phi'_{i-1}(\mu))$. We say that $\mu \in J(X)$ is an *invariant max-min measure* if $\Phi'_i(\mu) = \mu$ for every $i = 0, 1, \dots$ (equivalently, $\Phi'_1(\mu) = \mu$).

The following can be proved similarly as Theorem 1.

Theorem 3. *There exists a unique invariant max-min measure for the IFS f_1, \dots, f_n and $\alpha_1, \dots, \alpha_n \in \bar{\mathbb{R}}$ with $\bigoplus_{i=1}^n \alpha_i = \infty$. This invariant measure is the limit of the sequence $(\Phi'_i(\mu))_{i=1}^\infty$, for arbitrary $\mu \in J(X)$.*

The notion of inhomogeneous invariant max-plus measure can be defined similarly to that of inhomogeneous invariant idempotent measure. One can also formulate (and prove) a counterpart of Theorem 3 for the inhomogeneous invariant max-plus measures.

4 REMARKS AND OPEN QUESTIONS

Our construction is in a sense parallel to that of the invariant probability measure from [16]. The latter implicitly exploits the structure of monad for the probability measure functor P (more specifically, the so-called multiplication map $P^2 \rightarrow P$) and, in our case, the definition of Φ is based on the monad structure for the functor I (see [22]).

The proof of existence of the invariant probability measure implicitly uses the existence of a ‘nice’ functorial metrization of the spaces of probability measures of metric spaces. In particular, this metrization satisfies the property that the mentioned multiplication map $P^2(X) \rightarrow P(X)$ is nonexpanding and it is well-known that the Kantorovich metrization is as required [16,21]. Note that a metrization of the spaces $I(X)$ is constructed in [5]. However, it is not known whether the multiplication map $I^2(X) \rightarrow I(X)$ is non-expanding, for a metric space X . Taras Banach informed the authors that one can construct a metrization of the spaces $I(X)$ which allows for applying Banach’s contraction principle. As far as we know, his result is not published. Remark that the existence of invariant objects for IFSs in some general assumptions was considered in [3].

Some other generalizations can be made for the so called Lawson monads in the category **Comp** introduced by T. Radul [20].

Note that in [10] the first-named author considered the invariant inclusion hyperspaces for IFSs in complete metric spaces.

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Мазуренко Н., Зарічний М. *Інваріантні ідемпотентні міри* // Карпатські матем. публ. — 2018. — Т.10, №1. — С. 172–178.

Ідемпотентна математика є частиною математики, в якій арифметичні операції на множині дійсних чисел замінюються ідемпотентними операціями. У ідемпотентній математиці поняття ідемпотентної міри (міри Маслова) є відповідником поняття ймовірнісної міри. Ідемпотентні міри знайшли численні застосування в математиці та суміжних областях, зокрема, в теорії оптимізації, математичній морфології та теорії ігор.

У цій замітці ми запроваджуємо поняття інваріантної ідемпотентної міри для ітерованої системи функцій у повному метричному просторі. Це ідемпотентний аналог поняття інваріантної ймовірнісної міри, означеної Гатчінсоном. Зауважимо, що поняття інваріантної ідемпотентної міри раніше розглядалося авторами для класу ультраметричних просторів.

Одним з основних результатів є теорема існування та єдиності для інваріантних ідемпотентних мір у повних метричних просторах. На відміну від відповідного результату Гатчінсона для інваріантних ймовірнісних мір, наше доведення не опирається на метризацію простору ідемпотентних мір.

Аналогічний результат можна також довести для так званих неоднорідних ідемпотентних мір у повних метричних просторах.

Також наші міркування можна поширити на випадок max-min мір у повних метричних просторах.

Ключові слова і фрази: ідемпотентна міра (міра Маслова), система ітерованих відображень, інваріантна міра.



ROMANIV A.M.

ON THE STRUCTURE OF LEAST COMMON MULTIPLE MATRICES FROM SOME CLASS OF MATRICES

For non-singular matrices with some restrictions, we establish the relationships between Smith normal forms and transforming matrices (a invertible matrices that transform the matrix to its Smith normal form) of two matrices with corresponding matrices of their least common right multiple over a commutative principal ideal domains. Thus, for such a class of matrices, given answer to the well-known task of M. Newman. Moreover, for such matrices, received a new method for finding their least common right multiple which is based on the search for its Smith normal form and transforming matrices.

Key words and phrases: Smith normal form, transforming matrices, least common multiple matrices, commutative principal ideal domain .

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INTRODUCTION

Among the different problems and methods of their solutions that are considered in the commutative ring theory the special role is played by those that are similar to integer arithmetic ones, and they are the essential part of rings arithmetics. One of the mentioned problems, that are connected to the elementary divisibility theory, is how one can find the greatest common divisor and the least common multiple of given matrices over some ring and when such objects exist. The research in the area of such problems has started at the beginning of 20th century. Due in essence to E. Cahen and A. Chatelet, C. MacDuffee [4] has proposed elegant method of finding the greatest common divisor and the least common multiple of matrices using their Hermite forms. M. Newman and R. Thompson [10] studied the question: how to find the invariant multipliers of greatest common divisor and least common multiple of matrices over commutative principal ideal domains. The similar researches over the Euclidean domains became rather active in the recent years, as can be seen in the works of V. Nanda [5] , C. Yang, B. Li [1], S. Damkaew, S. Prugsapitak [2], N. Erawaty, M. Bahri, L. Haraynto, A. Amir [3] et al. In the current research author propose a method how to find least common multiple of matrices over commutative principal ideal domains, based on the properties of their Smith normal forms and the invertible matrices that transform these matrices to their Smith normal forms.

Let R be a commutative principal ideal domain with $1 \neq 0$, $M_n(R)$ be a ring $n \times n$ matrices over R . Consider a nonsingular matrix $A \in M_n(R)$. Since R is a principal ideal domain there are invertible matrices P_A, Q_A , such that

$$P_A A Q_A = E = \text{diag}(1, \varepsilon, \dots, \varepsilon).$$

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The matrix E above is called the Smith normal form or (canonical diagonal form) of matrix A , and matrices P_A and Q_A are left and right transforming matrices of A respectively.

By \mathbf{P}_A we denote the set of all left transforming matrices of matrix A . According to the results [8, 11] we know that $\mathbf{P}_A = \mathbf{G}_E P_A$, where

$$\mathbf{G}_E = \{H \in GL_n(R) \mid \exists H_1 \in GL_n(R) : HE = EH_1\}.$$

Note that it is a multiplicative group.

Suppose that the greatest common divisor of minor of size $n - 1$ of matrix B equals 1. Then

$$B \sim \Delta = \text{diag}(1, \dots, 1, \delta).$$

In the following we will use the set of matrices

$$\mathbf{L}(E, \Delta) = \{L \in GL_n(R) \mid \exists L_1 \in M_n(R) : LE = \Delta L_1\},$$

which is called a generating set (introduced by V. Shchedryk [8]).

If $A = BC$, then we will say that B is a left divisor of matrix A and A is a right multiple of B .

Moreover, if $M = AA_1 = BB_1$ then the matrix M is called a common right multiple of matrices A and B . If in addition the matrix M above is a left divisor of any other common right multiple of matrices A and B then we say that M is a **least common right multiple** of A and B . ($[A, B]_r$ in notation).

By the symbols (a, b) and $[a, b]$ we denote the greatest common divisor and the least common multiple of the elements a and b respectively, and the notation $a|b$ means that the element a divides the element b .

1 MAIN RESULTS

Lemma 1. *Let $P_B P_A^{-1} = S = \|s_{ij}\|_1^n$. Then the element $((\varepsilon, \delta), s_{n1})$ is an invariant with respect to transforming matrices P_B and P_A .*

Proof. Let $F_A \in \mathbf{P}_A$ and $F_B \in \mathbf{P}_B$ be some other left transforming matrices of A and B . Then exist matrices $H_A \in \mathbf{G}_E$ and $H_B \in \mathbf{G}_\Delta$ such that $F_A = H_A P_A$, $F_B = H_B P_B$. Consider the following product of the matrices:

$$F_B F_A^{-1} = H_B P_B (H_A P_A)^{-1} = H_B P_B P_A^{-1} H_A^{-1} = H_B S H_A^{-1},$$

where $S = P_B P_A^{-1}$. Let's denote $H_B S = \|k_{ij}\|_1^n$. In view of Corollary 6 [8] H_B is of the form

$$H_B = \left\| \begin{array}{cccc} h_{11} & \dots & h_{1,n-1} & h_{1n} \\ \dots & \dots & \dots & \dots \\ h_{n-1,1} & \dots & h_{n-1,n-1} & h_{n-1,n} \\ \delta h_{n1} & \dots & \delta h_{n,n-1} & h_{nn} \end{array} \right\|.$$

Hence,

$$\begin{aligned} k_{n1} &= \left\| \begin{array}{cccc} \delta h_{n1} & \dots & \delta h_{n,n-1} & h_{nn} \end{array} \right\| \left\| \begin{array}{c} s_{11} \\ \vdots \\ s_{n-1,1} \\ s_{n1} \end{array} \right\| \\ &= \delta(h_{n1}s_{11} + \dots + h_{n,n-1}s_{n-1,1}) + h_{nn}s_{n1} = \delta l + h_{nn}s_{n1}. \end{aligned}$$

Consider the following greatest common divisor:

$$((\varepsilon, \delta), k_{n1}) = ((\varepsilon, \delta), \delta l + h_{nn}s_{n1}) = ((\varepsilon, \delta), h_{nn}s_{n1}).$$

The invertibility of H_B implies that $(\delta, h_{nn}) = 1$. Therefore, $((\varepsilon, \delta), h_{nn}) = 1$ and

$$((\varepsilon, \delta), k_{n1}) = ((\varepsilon, \delta), s_{n1}).$$

Let's denote $SH_A^{-1} = \|t_{ij}\|_1^n$. Since $H_A^{-1} \in \mathbf{G}_E$ then according to Corollary 6 of [8] the matrix H_A^{-1} has the form

$$H_A^{-1} = \left\| \begin{array}{cccc} v_{11} & v_{12} & \dots & v_{1n} \\ \varepsilon v_{21} & v_{22} & \dots & v_{2n} \\ \dots & \dots & \dots & \dots \\ \varepsilon v_{n1} & v_{n2} & \dots & v_{nn} \end{array} \right\|.$$

Hence,

$$t_{n1} = \left\| \begin{array}{cccc} s_{n1} & s_{n2} & \dots & s_{nn} \end{array} \right\| \left\| \begin{array}{c} v_{11} \\ \varepsilon v_{21} \\ \vdots \\ \varepsilon v_{n1} \end{array} \right\| = s_{n1}v_{11} + \varepsilon(s_{n2}v_{21} + \dots + s_{nn}v_{n1}).$$

Consider

$$((\varepsilon, \delta), t_{n1}) = ((\varepsilon, \delta), s_{n1}v_{11} + \varepsilon(s_{n2}v_{21} + \dots + s_{nn}v_{n1})) = ((\varepsilon, \delta), s_{n1}v_{11}).$$

Since $(\varepsilon, v_{11}) = 1$, then $((\varepsilon, \delta), s_{n1}v_{11}) = ((\varepsilon, \delta), s_{n1})$. Hence

$$((\varepsilon, \delta), t_{n1}) = ((\varepsilon, \delta), s_{n1}).$$

Applying the associativity of $M_n(R)$ completes the proof. □

Lemma 2. Let $S = \|s_{ij}\|_1^n \in GL_n(R)$, $\Omega = \text{diag}(\omega_1, \omega_2, \dots, \omega_n)$, where $\omega_i \mid \omega_{i+1}$, $i = 1, 2, \dots, n - 1$, and $E \mid \Omega, \Delta \mid \Omega$. In order to $SL_A = L_B$, where $L_A \in \mathbf{L}(\Omega, E)$, $L_B \in \mathbf{L}(\Omega, \Delta)$ it is necessary and sufficient that $(a, b) \mid s_{n1}$, where $a = \frac{\varepsilon}{(\varepsilon, \omega_1)}$, $b = \frac{\delta}{(\delta, \omega_1)}$.

Proof. Necessity. Since $E \mid \Omega$ then according to Corollary 5 of [8] matrices L_A and L_B are of forms:

$$L_A = \left\| \begin{array}{cccc} p_{11} & p_{12} & \dots & p_{1n} \\ \frac{\varepsilon}{(\varepsilon, \omega_1)} p_{21} & p_{22} & \dots & p_{2n} \\ \vdots & \vdots & \dots & \vdots \\ \frac{\varepsilon}{(\varepsilon, \omega_1)} p_{n1} & p_{n2} & \dots & p_{nn} \end{array} \right\|, \quad L_B = \left\| \begin{array}{cccc} q_{11} & \dots & q_{1,n-1} & q_{1n} \\ q_{21} & \dots & q_{2,n-1} & q_{2n} \\ \vdots & \vdots & \dots & \vdots \\ \frac{\delta}{(\delta, \omega_1)} q_{n1} & \dots & \frac{\delta}{(\delta, \omega_{n-1})} q_{n-1,n-1} & q_{nn} \end{array} \right\|,$$

respectively. Using the Property 4.8 [9], in this case, the set $\mathbf{L}(\Omega, E)$ is a group. Then $S = L_B L_A^{-1}$, where $L_A^{-1} \in \mathbf{L}(\Omega, E)$. It follows that

$$\left(\frac{\varepsilon}{(\varepsilon, \omega_1)}, \frac{\delta}{(\delta, \omega_1)} \right) \mid s_{n1}.$$

If we denote $a = \frac{\varepsilon}{(\varepsilon, \omega_1)}$, $b = \frac{\delta}{(\delta, \omega_1)}$ then we will get that $(a, b) \mid s_{n1}$.

Sufficiency. Let $s_{n1} = (a, b)t$. By Theorem 2.13 [9] there exist some matrices $H_1 \in \mathbf{G}_\Delta$ and $U \in \mathbf{G}_E$ such that

$$H_1SU = \left\| \begin{array}{cccccc} 1 & 0 & \dots & 0 & 0 \\ k_{21} & 1 & \dots & 0 & 0 \\ \vdots & \dots & \ddots & \vdots & \vdots \\ k_{n-1,1} & k_{n-1,2} & \dots & 1 & 0 \\ (a,b)k_{n1} & k_{n2} & \dots & k_{n,n-1} & 1 \end{array} \right\| = \left\| \begin{array}{cc} K_{11} & \mathbf{0} \\ K_{21} & 1 \end{array} \right\|.$$

Obviously, K_{11} is invertible. Hence there exists some matrix $H_2 = \left\| \begin{array}{cc} K_{11}^{-1} & \mathbf{0} \\ \mathbf{0} & 1 \end{array} \right\| \in \mathbf{G}_\Delta$ such that

$$H_2H_1SU = \left\| \begin{array}{cccccc} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \dots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ (a,b)k_{n1} & k_{n2} & \dots & k_{n,n-1} & 1 \end{array} \right\| = K.$$

Since $H_1, H_2 \in \mathbf{G}_\Delta$ then $H_3 = H_2H_1 \in \mathbf{G}_\Delta$. Therefore $K = H_3SU$. Moreover, one can find $v_1, v_2 \in R$ such that

$$(a,b)k_{n1} = (av_1 + bv_2)k_{n1} = av_1k_{n1} + bv_2k_{n1}.$$

If we consider the matrices

$$H_4 = \left\| \begin{array}{ccccc} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \\ bv_2k_{n1} & 0 & \dots & 0 & 1 \end{array} \right\| \in \mathbf{L}(\Omega, \Delta)$$

and

$$V = \left\| \begin{array}{ccccc} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \\ -av_1k_{n1} & -k_{n2} & \dots & -k_{n,n-1} & 1 \end{array} \right\| \in \mathbf{L}(\Omega, E).$$

we obtain that $H_3SUV = H_4$. Then $SUV = H_3^{-1}H_4$. Using Properties 2 and 3 [8] we will have $H_3^{-1}H_4 = L_B \in \mathbf{L}(\Omega, \Delta)$, $UV = L_A \in \mathbf{L}(\Omega, E)$, and so $SL_A = L_B$ which had to be proved. \square

Theorem 1. Let R be a commutative principal ideal domain and let

$$A \sim \text{diag}(1, \varepsilon, \dots, \varepsilon), \quad B \sim \text{diag}(1, \dots, 1, \delta),$$

$P_B P_A^{-1} = \|s_{ij}\|_1^n$, $P_B \in \mathbf{P}_B$, $P_A \in \mathbf{P}_A$. Then

$$[A, B]_r = (L_A P_A)^{-1} \Omega = (L_B P_B)^{-1} \Omega,$$

where

$$\Omega = \text{diag}\left(\frac{(\varepsilon, \delta)}{((\varepsilon, \delta), s_{n1})}, \varepsilon, \dots, \varepsilon, [\varepsilon, \delta]\right),$$

L_A, L_B belong to sets $\mathbf{L}(\Omega, E)$, $\mathbf{L}(\Omega, \Delta)$ respectively and satisfy the equality:

$$(P_B P_A^{-1}) L_A = L_B.$$

Proof. Remark that according to Lemma (1), the element $((\varepsilon, \delta), s_{n1})$, and hence the matrix Ω , does not depend on the choice of transforming matrices P_A and P_B .

By Theorem 2 [6] the Smith normal form of the greatest common left divisor of the matrices A and B is of the form

$$(A, B)_l \sim \text{diag}(1, \dots, 1, (\varepsilon, \delta, s_{n1})).$$

According to Corollary 1.5 [9] we obtain

$$\pm \det A \det B = \det(A, B)_l \det[A, B]_r,$$

i.e.

$$\det[A, B]_r = \pm \frac{\det A \det B}{\det(A, B)_l} = \pm \frac{\varepsilon^{n-1} \delta}{(\varepsilon, \delta, s_{n1})} = \omega_1 \omega_2 \dots \omega_{n-1} \omega_n.$$

It follows from [10] that $\omega_n = [\varepsilon, \delta]$ and $\omega_i \mid \varepsilon, i = 2, \dots, n - 1$. Since $E \mid \Omega$ then $\varepsilon \mid \omega_i$, for $i = 2, \dots, n - 1$, that is $\omega_i = \varepsilon, i = 2, \dots, n - 1$. Hence,

$$\omega_1 = \pm \frac{\varepsilon^{n-1} \delta (\varepsilon, \delta)}{\varepsilon^{n-2} \varepsilon \delta (\varepsilon, \delta, s_{n1})} = \pm \frac{(\varepsilon, \delta)}{(\varepsilon, \delta, s_{n1})}.$$

Taking into account that the invariant factors of matrix are chosen precisely to the divisors of unit, we obtain that the Smith normal form of the least common right multiple of matrices A and B has the form:

$$\Omega = \text{diag}\left(\frac{(\varepsilon, \delta)}{((\varepsilon, \delta), s_{n1})}, \varepsilon, \dots, \varepsilon, [\varepsilon, \delta]\right).$$

By Lemma 1 [7] we will have

$$\left(\frac{\varepsilon}{(\varepsilon, \omega_1)}, \frac{\delta}{(\delta, \omega_1)}\right) = \left(\frac{(\varepsilon, \delta)}{((\varepsilon, \delta), \omega_1)}\right) = \mu.$$

Since $\omega_1 = \frac{(\varepsilon, \delta)}{(\varepsilon, \delta, s_{n1})}$, then

$$\mu = \frac{(\varepsilon, \delta)}{((\varepsilon, \delta), \frac{(\varepsilon, \delta)}{(\varepsilon, \delta, s_{n1})})} = \frac{(\varepsilon, \delta)((\varepsilon, \delta), s_{n1})}{((\varepsilon, \delta)((\varepsilon, \delta), s_{n1}), (\varepsilon, \delta))} = ((\varepsilon, \delta), s_{n1}).$$

This means that $\mu \mid s_{n1}$. According to Lemma (2) there exist matrices $L_A \in \mathbf{L}(\Omega, E)$, $L_B \in \mathbf{L}(\Omega, \Delta)$ such that $P_B P_A^{-1} L_A = L_B$, so

$$P_A^{-1} L_A \Omega = P_B^{-1} L_B \Omega = M.$$

Since $E \mid \Omega$ and $\Delta \mid \Omega$, then using Theorem 1 [8] the matrix M is the common right multiple of A and B .

Let N be least common right multiple of matrices A and B . From the above, it follows that $N \sim \Omega$. Hence $N = P_N^{-1} \Omega Q_N^{-1}$. Then $M = P_A^{-1} L_A \Omega = P_M^{-1} \Omega$ is a right multiple of N : $M = N N_1$. According to Theorem 1 [8] this is equivalent to the fact that $P_N = L P_M$, where $L \in \mathbf{L}(\Omega, \Omega)$. Using Property 4.6 [9] we get the equality $\mathbf{L}(\Omega, \Omega) = \mathbf{G}_\Omega$. Then by Corollary 2 [8] the matrices M and N are right associated. Thus, M is the least common right multiples of matrices A and B . The theorem is proved. \square

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Для неособливих матриць, при певних обмеженнях, встановлено взаємозв'язки між формами Сміта та перетворювальними матрицями (оборотними матрицями, що зводять матрицю до її форми Сміта) двох матриць з відповідними матрицями їх найменшого спільного правого кратного над комутативними областями головних ідеалів. Тим самим, для такого класу матриць, дано відповідь на відому задачу М. Ньюмена. Більше того, для таких матриць, вказано новий метод знаходження їх найменшого спільного правого кратного, яких ґрунтується на пошуку його форми Сміта та перетворювальних матриць.

Ключові слова і фрази: форма Сміта, перетворювальні матриці, найменше спільне кратне матриць, комутативна область головних ідеалів.

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SIGNLESS LAPLACIAN DETERMINATIONS OF SOME GRAPHS WITH INDEPENDENT EDGES

Let G be a simple undirected graph. Then the signless Laplacian matrix of G is defined as $D_G + A_G$ in which D_G and A_G denote the degree matrix and the adjacency matrix of G , respectively. The graph G is said to be determined by its signless Laplacian spectrum (DQS, for short), if any graph having the same signless Laplacian spectrum as G is isomorphic to G . We show that $G \sqcup rK_2$ is determined by its signless Laplacian spectra under certain conditions, where r and K_2 denote a natural number and the complete graph on two vertices, respectively. Applying these results, some DQS graphs with independent edges are obtained.

Key words and phrases: spectral characterization, signless Laplacian spectrum, cospectral graphs, union of graphs.

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INTRODUCTION

All graphs considered here are simple and undirected. All notions on graphs that are not defined here can be found in [13, 16]. Let G be a simple graph with the vertex set $V = V(G) = \{v_1, \dots, v_n\}$ and the edge set $E = E(G)$. Denote by d_i the degree of the vertex v_i . The *adjacency matrix* A_G of G is a square matrix of order n , whose (i, j) -entry is 1 if v_i and v_j are adjacent in G and 0 otherwise. The *degree matrix* D_G of G is a diagonal matrix of order n defined as $D_G = \text{diag}(d_1, \dots, d_n)$. The matrices $L_G = D_G - A_G$ and $Q_G = D_G + A_G$ are called the *Laplacian matrix* and the *signless Laplacian matrix* of G , respectively. The multiset of eigenvalues of Q_G (resp. L_G , A_G) is called the *Q-spectrum* (resp. *L-spectrum*, *A-spectrum*) of G . For any bipartite graph, its *Q-spectrum* coincides with its *L-spectrum*. Two graphs are *Q-cospectral* (resp. *L-cospectral*, *A-cospectral*) if they have the same *Q-spectrum* (resp. *L-spectrum*, *A-spectrum*). A graph G is said to be *DQS* (resp. *DLS*, *DAS*) if there is no other non-isomorphic graph *Q-cospectral* (resp. *L-cospectral*, *A-cospectral*) with G . Let us denote the *Q-spectrum* of G by $\text{Spec}_Q(G) = \{[q_1]^{m_1}, [q_2]^{m_2}, \dots, [q_n]^{m_n}\}$, where m_i denotes the multiplicity of q_i and $q_1 \geq q_2 \geq \dots \geq q_n$.

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The *join* of two graphs G and H is a graph formed from disjoint copies of G and H by connecting each vertex of G to each vertex of H . We denote the join of two graphs G and H by $G \nabla H$. The complement of a graph G is denoted by \overline{G} . For two disjoint graphs G and H , let $G \sqcup H$ denotes the disjoint union of G and H , and rG denotes the disjoint union of r copies of G , i.e., $rG = \underbrace{G \sqcup \dots \sqcup G}_{r\text{-times}}$.

Let G be a connected graph with n vertices and m edges. Then G is called *unicyclic* (resp. *bicyclic*) if $m = n$ (resp. $m = n + 1$). If G is a unicyclic graph containing an odd (resp. even) cycle, then G is called *odd unicyclic* (resp. *even unicyclic*).

Let C_n, P_n, K_n be the cycle, the path and the complete graph of order n , respectively. $K_{s,t}$ the complete bipartite graph with s vertices in one part and t in the other.

Let us remind that the *coalescence* [21] of two graphs G_1 with distinguished vertex v_1 and G_2 with distinguished vertex v_2 , is formed by identifying vertices v_1 and v_2 that is, the vertices v_1 and v_2 are replaced by a single vertex v adjacent to the same vertices in G_1 as v_1 and the same vertices in G_2 as v_2 . If it is not necessary v_1 or v_2 may not be specified.

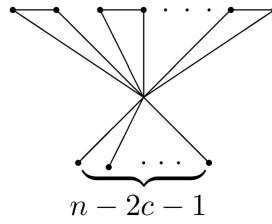
The *friendship graph* F_n is a graph with $2n + 1$ vertices and $3n$ edges obtained by the coalescence of n copies of C_3 with a common vertex as the distinguished vertex; in fact, F_n is nothing but $K_1 \nabla nK_2$.

The *lollipop graph*, denoted by $H_{n,p}$, is the coalescence of a cycle C_p with arbitrary distinguished vertex and a path P_{n-p} with a pendent vertex as the distinguished vertex; for example $H_{11,6}$ is depicted in Figure 1 (b). We denote by $T(a, b, c)$ the *T-shape tree* obtained by identifying the end vertices of three paths P_{a+2}, P_{b+2} and P_{c+2} . In fact, $T(a, b, c)$ is a tree with one and only one vertex v of degree 3 such that $T(a, b, c) - \{v\} = P_{a+1} \sqcup P_{b+1} \sqcup P_{c+1}$; for example $T(0, 1, 1)$ is depicted in Figure 1 (a).

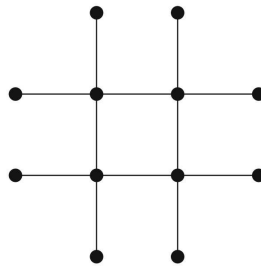


Figure 1: (a) The T-shape tree $T(0, 1, 1)$ (b) The lollipop graph $H_{11,6}$

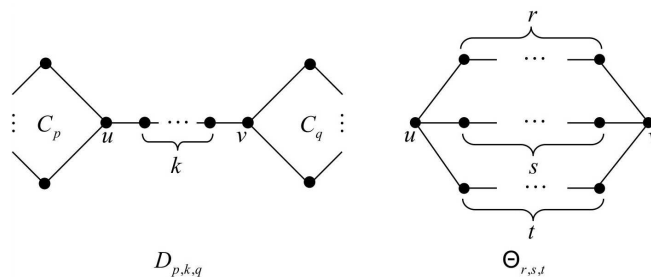
A *kite graph* $Ki_{n,w}$ is a graph obtained from a clique K_w and a path P_{n-w} is the coalescence of K_w with an arbitrary distinguished vertex and a path P_{n-w+1} with a pendent vertex as the distinguished vertex. A tree is called *starlike* if it has exactly one vertex of degree greater than two. We denote by $U_{r,n-r}$ the graph obtained by attaching $n - r$ pendent vertices to a vertex of C_r . In fact, $U_{r,n-r}$ is the coalescence of $K_{1,n-r-1}$ and P_{n-w+1} where distinguished vertices are the vertex of degree $n - r$ and a pendent vertex, respectively. A graph is a *cactus*, or a *treelike graph*, if any pair of its cycles has at most one common vertex [35]. If all cycles of the cactus G have exactly one common vertex, then G is called a *bundle* [12]. Let $S(n, c)$ be the bundle with n vertices and c cycles of length 3 depicted in Figure 2, where $n \geq 2c + 1$ and $c \geq 0$. By the definition, it follows that $S(n, c) = K_1 \nabla (cK_2 \sqcup (n - 2c - 1)K_1)$. In fact $S(n, c)$ is the coalescence of F_c and $K_{1,n-2c-1}$ where the distinguished vertices are the vertex of the degree $2c$ and the vertex of the degree $n - 2c - 1$, respectively.

Figure 2: The bundle $S(n, c)$

Let G be a graph with n vertices, H be a graph with m vertices. The *corona* of G and H , denoted by $G \circ H$, is the graph with $n + mn$ vertices obtained from G and n copies of H by joining the i -th vertex of G to each vertex in the i -th copy of H ($i \in \{1, \dots, n\}$); for example $C_4 \circ 2K_1$ is depicted in Figure 3.

Figure 3: $C_4 \circ 2K_1$

A *complete split graph* $CS(n, \alpha)$, is a graph on n vertices consisting of a clique on $n - \alpha$ vertices and an independent set on the remaining α ($1 \leq \alpha \leq n - 1$) vertices in which each vertex of the clique is adjacent to each vertex of the independent set. The *dumbbell graph*, denoted by $D_{p,k,q}$, is a bicyclic graph obtained from two cycles C_p, C_q and a path P_{k+2} by identifying each pendant vertex of P_{k+2} with a vertex of a cycle, respectively. The *theta graph*, denoted by $\Theta_{r,s,t}$, is the graph formed by joining two given vertices via three disjoint paths P_r, P_s and P_t , respectively, see Figure 4.

Figure 4: The graphs $D_{p,k,q}$ and $\Theta_{r,s,t}$

The problem "which graphs are determined by their spectrum?" was posed by Günthard and Primas [24] more than 60 years ago in the context of Hückel's theory in chemistry. In the most recent years mathematicians have devoted their attention to this problem and many

papers focusing on this topic are now appearing. In [36] van Dam and Haemers conjectured that almost all graphs are determined by their spectra. Nevertheless, the set of graphs that are known to be determined by their spectra is too small. So, discovering infinite classes of graphs that are determined by their spectra can be an interesting problem. Cvetković, Rowlinson and Simić in [17–20] discussed the development of a spectral theory of graphs based on the signless Laplacian matrix, and gave several reasons why it is superior to other graph matrices such as the adjacency and the Laplacian matrix. It is interesting to construct new DQS (DLS) graphs from known DQS (DLS) graphs. Up to now, only some graphs with special structures are shown to be *determined by their spectra* (DS, for short) (see [1–11, 15, 17, 19, 22, 23, 25–34, 38–41] and the references cited in them). About the background of the question “Which graphs are determined by their spectrum?”, we refer to [36, 37]. For a DQS graph G , $G \nabla K_2$ is also DQS under some conditions [30]. A graph is DLS if and only if its complement is DLS. Hence we can obtain DLS graphs from known DLS graphs by adding independent edges. In [25] it was shown that $G \sqcup rK_1$ is DQS under certain conditions. In this paper, we investigate signless Laplacian spectral characterization of graphs with independent edges. For a DQS graph G , we show that $G \sqcup rK_2$ is DQS under certain conditions. Applying these results, some DQS graphs with independent edges are obtained.

1 PRELIMINARIES

In this section, we give some lemmas which are used to prove our main results.

Lemma 1 ([17, 19]). *Let G be a graph. For the adjacency matrix of G , the following can be deduced from the spectrum.*

- (1) *The number of vertices.*
- (2) *The number of edges.*
- (3) *Whether G is regular.*

For the Laplacian matrix, the following follows from the spectrum:

- (4) *The number of components.*

For the signless Laplacian matrix, the following follow from the spectrum:

- (5) *The number of bipartite components, i.e., the multiplicity of the eigenvalue 0 of the signless Laplacian matrix is equal to the number of bipartite components.*
- (6) *The sum of the squares of degrees of vertices.*

Lemma 2 ([17]). *Let G be a graph with n vertices, m edges, t triangles and the vertex degrees d_1, d_2, \dots, d_n . If $T_k = \sum_{i=1}^n q_i(G)^k$, then we have*

$$T_0 = n, \quad T_1 = \sum_{i=1}^n d_i = 2m, \quad T_2 = 2m + \sum_{i=1}^n d_i^2, \quad T_3 = 6t + 3 \sum_{i=1}^n d_i^2 + \sum_{i=1}^n d_i^3.$$

For a graph G , let $P_L(G)$ and $P_Q(G)$ denote the product of all nonzero eigenvalues of L_G and Q_G , respectively. Note that $P_L(K_2) = P_Q(K_2) = 2$. We assume that $P_L(G) = P_Q(G) = 1$ if G has no edges.

Lemma 3 ([16]). *For any connected bipartite graph G of order n , we have $P_Q(G) = P_L(G) = n\tau(G)$, where $\tau(G)$ is the number of spanning trees of G . Especially, if T is a tree of order n , then $P_Q(T) = P_L(T) = n$.*

Lemma 4 ([32]). *Let G be a graph with n vertices and m edges.*

- (i) $\det(Q_G) = 4$ if and only if G is an odd unicyclic graph.
- (ii) If G is a non-bipartite connected graph and $m > n$, then $\det(Q_G) \geq 16$, with equality if and only if G is a non-bipartite bicyclic graph with C_4 as its induced subgraph.

Lemma 5 ([16]). *Let e be any edge of a graph G of order n . Then*

$$q_1(G) \geq q_1(G - e) \geq q_2(G) \geq q_2(G - e) \geq \dots \geq q_n(G) \geq q_n(G - e) \geq 0.$$

Lemma 6 ([21]). *Let H be a proper subgraph of a connected graph G . Then $q_1(G) > q_1(H)$.*

Lemma 7 ([21]). *Let G be a graph with n vertices and m edges. Then $q_1(G) \geq \frac{4m}{n}$, with equality if and only if G is regular.*

Lemma 8 ([17]). *For a graph G , $0 < q_1(G) < 4$ if and only if all components of G are paths.*

Lemma 9 ([36]). *A regular graph is DQS if and only if it is DAS. A regular graph G is DAS (DQS) if and only if \overline{G} is DAS (DQS).*

Lemma 10 ([19]). *Let G be a k -regular graph of order n . Then G is DAS when $k \in \{0, 1, 2, n - 3, n - 2, n - 1\}$.*

Lemma 11 ([15]). *Let G be a k -regular graph of order n . Then $G \nabla K_1$ is DQS for $k \in \{1, n - 2\}$, for $k = 2$ and $n \geq 11$. For $k = n - 3$, $G \nabla K_1$ is DQS if and only if \overline{G} has no triangles.*

Lemma 12 ([30]). *Let G be a k -regular graph of order n . Then $G \nabla K_2$ is DQS for $k \in \{1, n - 2\}$. For $k = n - 3$, $G \nabla K_2$ is DQS if and only if \overline{G} has no triangles.*

Lemma 13 ([25]). *The following hold for graphs with isolated vertices:*

- (i) Let T be a DLS tree of order n . Then $T \sqcup rK_1$ is DLS. If n is not divisible by 4, then $T \sqcup rK_1$ is DQS.
- (ii) The graphs $\overline{P_n}$ and $\overline{P_n} \sqcup rK_1$ are DQS.
- (iii) Let G be a graph obtained from K_n by deleting a matching. Then G and $G \sqcup rK_1$ are DQS.
- (iv) A $(n - 4)$ -regular graph of order n is DAS (DQS) if and only if its complement is a 3-regular DAS (DQS) graph.
- (v) Let G be a $(n - 3)$ -regular graph of order n . Then $G \sqcup rK_1$ is DQS.

Now let us list some known families of DQS graphs.

Lemma 14. *The following graphs are DQS.*

- (i) *The graphs $P_n, C_n, K_n, K_{m,m}, rK_n, P_{n_1} \sqcup P_{n_2} \sqcup \dots \sqcup P_{n_k}$ and $C_{n_1} \sqcup C_{n_2} \sqcup \dots \sqcup C_{n_k}$, [36].*
- (ii) *Any wheel graph $K_1 \nabla C_n$, [26].*
- (iii) *Every lollipop graph $H_{n,p}$, [41].*
- (iv) *Every kite graph $Ki_{n,n-1}$ for $n \geq 4$ and $n \neq 5$, [23].*
- (v) *The friendship graph F_n , [38].*
- (vi) *$(C_n \circ tK_1)$, for $n \notin \{32, 64\}$ and $t \in \{1, 2\}$, [14, 32].*
- (vii) *The line graph of a T-shape tree $T(a, b, c)$ except $T(t, t, 2t + 1)$ ($t > 1$), [39].*
- (viii) *The starlike tree with maximum degree 4, [34].*
- (ix) *$U_{r,n-r}$ for $r \geq 3$, [27].*
- (x) *$CS(n, \alpha)$ when $1 \leq \alpha \leq n - 1$ and $\alpha \neq 3$, [22].*
- (xi) *For $n \geq 2c + 1$ and $c \geq 0$, $\overline{S(n, c)}$ and $S(n, c)$ except for the case of $c = 0$ and $n = 4$, [29].*
- (xii) *$K_{1,n-1}$ for $n \neq 4$, [29].*
- (xiii) *$G \nabla K_m$ where G is an $(n - 2)$ -regular graph on n vertices, and $\overline{K_n} \nabla K_2$ except for $n = 3$, [28].*
- (xiv) *All dumbbell graphs different from $D_{3q,0,q}$ and all theta graphs, [40].*

It is easy to see that $K_{1,3}$ and $K_3 \sqcup K_1$ are Q -cospectral, i.e., $\text{Spec}_Q(K_{1,3}) = \text{Spec}_Q(K_3) = \{[4]^1, [1]^2, [0]^1\}$. Therefore, $S(n, c)$ is not DQS when $c = 0$ and $n = 4$, since $S(n, 0)$ is nothing but $K_{1,n-1}$.

2 MAIN RESULTS

We first investigate spectral characterizations of the union of a tree and several complete graphs K_2 .

Theorem 1. *Let T be a DLS tree of order n . Then $T \sqcup rK_2$ is DLS for any positive integer r . Moreover, if n is odd and $r = 1$, then $T \sqcup rK_2$ is DQS.*

Proof. For $n, r \in \{1, 2\}$ see Lemma 13 (i) and Lemma 14 (i). So, one may suppose that $n, r \geq 3$. Let G be any graph L -cospectral with $T \sqcup rK_2$. By Lemma 1, G has $n + 2r$ vertices, $n - 1 + r$ edges and $r + 1$ components. So each component of G is a tree. Suppose that $G = G_0 \sqcup G_1 \sqcup \dots \sqcup G_r$, where G_i is a tree with n_i vertices and $n_0 \geq n_1 \geq \dots \geq n_r \geq 2$. For $n_i, n_r \in \{1\}$ see Lemma 13 (i) and Lemma 14 (i). Hence we consider $n, n_i, r \geq 2$. Since G is L -cospectral with $T \sqcup rK_2$, by Lemma 3, we get

$$n_0 n_1 \dots n_r = P_L(G_0) \dots P_L(G_r) = P_L(G_0 \sqcup \dots \sqcup G_r) = P_L(G) = P_L(T) P_L(K_2)^r = n 2^r.$$

We claim that $n_r = 2$. Suppose not and so $n_r \geq 3$. This means that $n_0 \geq n_1 \geq \dots \geq n_r \geq 3$. Hence $n2^r = n_0 n_1 \dots n_r \geq 3^{r+1}$ or $n(\frac{2}{3})^r \geq 3$. Now, if $r \rightarrow \infty$, then $0 \geq 3$, a contradiction. So, we must have $n_r = 2$. By a similar argument one can show that $n_1 = \dots = n_{r-1} = 2$ and so $n_0 = n$. Hence $G = G_0 \sqcup rK_2$. Since G and $T \sqcup rK_2$ are L -cospectral, G_0 and T are L -cospectral. Since T is DLS, we have $G_0 = T$, and thus $G = T \sqcup rK_2$. Hence $T \sqcup rK_2$ is DLS.

Let H be any graph Q -cospectral with $T \sqcup rK_2$. By Lemma 1, H has $n + 2r$ vertices, $n - 1 + r$ edges and $r + 1$ bipartite components. So one of the following holds:

- (i) H has exactly $r + 1$ components, and each component of H is a tree.
- (ii) H has $r + 1$ components which are trees, the other components of H are odd unicyclic.

In what follows we show that (ii) does not occur if n is odd and $r = 1$. If (ii) holds, then by Lemma 4, $P_Q(H)$ is divisible by 4 since H has a cycle of odd order as a component. Since T is a tree of order n , by Lemma 3, $P_Q(H) = P_Q(T)P_Q(K_2)^r = n2^r$ is divisible by 4, a contradiction. Therefore (i) must hold. In this case, H and $T \sqcup rK_2$ are both bipartite, and so they are also L -cospectral. By the previous part, $T \sqcup rK_2$ is DLS. So we have $H = T \sqcup rK_2$.

Hence $T \sqcup rK_2$ is DQS when n is odd and $r = 1$. \square

Remark 1. Some DLS trees are given in [25] and references therein. We can obtain some DLS (DQS) trees with independent edges from Theorem 1.

Lemma 14 and Theorem 1 imply the following corollary.

Corollary 1. For an odd positive integer n , we have the following

- (i) Let T be a starlike tree of order n and with maximum degree 4. Then $T \sqcup K_2$ is DQS.
- (ii) $P_n \sqcup K_2$ is DQS.
- (iii) For $n \neq 4$, $K_{1,n-1} \sqcup K_2$ is DQS.
- (iv) Let \mathcal{L} be the line graph of a T -shape tree $T(a, b, c)$ except $T(t, t, 2t + 1)$ ($t > 1$). Then $\mathcal{L} \sqcup K_2$ is DQS if $a + b + c - 3$ is odd.

Theorem 2. Let G be a DQS odd unicyclic graph of order $n \geq 7$. Then $G \sqcup rK_2$ is DQS for any positive integer r .

Proof. Let H be any graph Q -cospectral with $G \sqcup rK_2$. By Lemma 1(5), 0 is not an eigenvalue of G since it is an odd unicyclic. So by Lemma 4, we have $4 = \det(Q_G) = P_Q(G)$. Moreover,

$$P_Q(H) = P_Q(G \sqcup rK_2) = P_Q(G)P_Q(K_2)^r = \det(Q_G)2^r = 4 \cdot 2^r = 2^{r+2}.$$

By Lemma 1, H has $n + 2r$ vertices, $n + r$ edges and r bipartite components. So one of the following holds:

- (i) H has exactly r components each of which is a tree.
- (ii) H has r components which are trees, the other components of H are odd unicyclic.

We claim that (i) does not hold, otherwise, we may assume that $H = H_1 \sqcup \dots \sqcup H_r$, where H_i is a tree with n_i vertices and $n_1 \geq \dots \geq n_r \geq 1$. It follows from Lemma 3 that

$$n_1 \dots n_r = P_Q(H_1) \dots P_Q(H_r) = P_Q(H) = 4 \cdot 2^r = 2^{r+2}.$$

So $n_1 \dots n_r = 2^{r+2}$, $n_1 \leq 8$. Since G contains a cycle, say C , by Lemma 7 we have

$$q_1(H) = q_1(G) \geq q_1(C) = 4. \tag{1}$$

Let $\Delta(H)$ be the maximum degree of H . If $\Delta(H) \leq 2$, then all components of H are paths, hence by Lemma 8, $q_1(H) < 4$, contradicting Eq. (1). So $\Delta(H) \geq 3$. From $n_1 \leq 8$ and $n_1 \dots n_r = 4 \cdot 2^r = 2^{(r+2)}$, we may assume that $H_1 = K_{1,7}$, $H_2 = \dots = H_r = K_2$. Since $H = K_{1,7} \sqcup (r-1)K_2$ has $n + 2r$ vertices, we get $n = 6$, a contradiction to $n \geq 7$.

If (ii) holds, then we may assume that $H = U_1 \sqcup \dots \sqcup U_c \sqcup H_1 \sqcup \dots \sqcup H_r$, where U_i is odd unicyclic, H_i is a tree with n_i vertices. By Lemmas 3 and 4, $4 \cdot 2^r = P_Q(H) = 4^c n_1 \dots n_r$. So $c = 1$, $H_1 = \dots = H_r = K_2$. Since $H = U_1 \sqcup rK_2$ and $G \sqcup rK_2$ are Q -cospectral, U_1 and G are Q -cospectral. Since G is DQS, we have $U_1 = G$, $H = G \sqcup rK_2$. \square

Remark 2. Note that $C_4 \sqcup 2P_3$ and $C_6 \sqcup 2K_2$ are Q -cospectral, i.e., $\text{Spec}_Q(C_4 \sqcup 2P_3) = \text{Spec}_Q(C_6 \sqcup 2K_2) = \{[4]^1, [3]^2, [2]^2, [1]^2, [0]^3\}$. It follows that the condition "odd unicyclic of order $n \geq 7$ " is essential in Theorem 2.

Remark 3. Some DQS unicyclic graphs are given in [25] and references therein. We can obtain some DQS graphs with independent edges from Theorem 2.

Theorem 3. Let G be a DQS graph of order $n \geq 5$. If G is non-bipartite bicyclic graph with C_4 as its induced subgraph, then $G \sqcup rK_2$ is DQS for any positive integer r .

Proof. Let H be any graph Q -cospectral with $G \sqcup rK_2$. By Lemma 4, we have

$$P_Q(H) = P_Q(G \sqcup rK_2) = P_Q(G)P_Q(K_2)^r = P_Q(G)2^r.$$

By Lemma 1(5), 0 is not an eigenvalue of G since it is non-bipartite. So by Lemma 4, we have $16 = \det(G_Q) = P_Q(G)$ and thus $P_Q(H) = 16 \cdot 2^r$.

By Lemma 1, H has $n + 2r$ vertices, $n + 1 + r$ edges and r bipartite components. So H has at least $r - 1$ components which are trees. Suppose that H_1, H_2, \dots, H_r are r bipartite components of H , where H_2, \dots, H_r are trees. If H_1 contains an even cycle, then by Lemmas 4 and 5, we have $P_Q(H) \geq P_Q(H_1) \geq 16$, and $P_Q(H) = 16 \cdot (2^{r-1}) = 2^{r-3}$ if and only if $H = C_4 \sqcup (r-1)K_2$. By $P_Q(H) = 16 \cdot (2^{r-1}) = 2^{r-3}$, we have $H = C_4 \sqcup (r-1)K_2$. Since H has $n + 2r$ vertices, we get $n = 2$, a contradiction (G contains C_4). Hence H_1, H_2, \dots, H_r are trees. Since H has $n + 2r$ vertices, $n + 1 + r$ edges and r bipartite components, H has a non-bipartite component H_0 which is a bicyclic graph. Lemmas 4 and 5 imply that $P_Q(H) \geq P_Q(H_0) \geq 16$, and $P_Q(H) = 16 \cdot 2^r$ if and only if $H = H_0 \sqcup rK_2$ and H_0 contains C_4 as its induced subgraph. By $P_Q(H) = 16 \cdot 2^r$, we have $H = H_0 \sqcup rK_2$. Since H and $G \sqcup rK_2$ are Q -cospectral, H_0 and G are Q -cospectral. Taking into account that G is DQS, we conclude that $H_0 = G$ and $H = G \sqcup rK_2$. Hence $G \sqcup rK_2$ is DQS. \square

Remark 4. Some DQS bicyclic graphs are given in [25] and references therein. We can obtain DQS graphs with independent edges from Theorem 3.

Lemma 15. *Let G be a connected graph. Then there is no subgraph of G with the Q -spectrum identical to $\text{Spec}_Q(G) \cup \{[2]^1\}$. Moreover, if G is of order at least 3, then $q_1(G) \geq 3$.*

Proof. Suppose by the contrary that there is a subgraph of G , say G' , such that $\text{Spec}_Q(G') = \text{Spec}_Q(G) \cup \{[2]^1\}$. But, in this case $|E(G')| = |E(G)| + 1$ and $|V(G')| = |V(G)| + 1$. Therefore there exists a vertex v of G' with the degree one such that $G' - v = G$. This means that G is a proper subgraph of the connected graph G' and so by Lemma 6, $q_1(G') > q_1(G)$, a contradiction. If G is a connected graph of order at least 3, it has K_3 or $K_{1,2}$ as its subgraph. Moreover, $\text{Spec}_Q(K_3) = \{[4], [1]^2\}$ and $\text{Spec}_Q(K_{1,2}) = \{[3], [1], [0]\}$. Therefore by Lemma 5, $q_1(G) \geq 3$. \square

Theorem 4. *Let G be a connected non-bipartite graph with $n \geq 3$ vertices which is DQS. Then for any positive integer r , $G \sqcup rK_2$ is DQS.*

Proof. Let H be a graph Q -cospectral with $G \sqcup rK_2$. Then by Lemmas 1 and 2, H has $n + 2r$ vertices, $n + 1 + r$ edges and exactly r bipartite components. We perform mathematical induction on r . Suppose that H is a graph Q -cospectral with $G \sqcup K_2$. Then

$$\text{Spec}_Q(H) = \text{Spec}_Q(G) \cup \text{Spec}_Q(K_2) = \text{Spec}_Q(G) \cup \{[2]^1, [0]^1\}.$$

Since G is a connected non-bipartite graph, by Lemma 1, it has not 0 as its signless Laplacian eigenvalue. Therefore, H has exactly one bipartite component. Therefore, by Lemma 15 we get $H = G \sqcup K_2$. Now, let the assertion holds for r ; that is, if $\text{Spec}_Q(G_1) = \text{Spec}_Q(G) \cup \text{Spec}_Q(rK_2)$, then $G_1 = G \sqcup rK_2$. We show that it follows from $\text{Spec}_Q(K) = \text{Spec}_Q(G) \cup \text{Spec}_Q((r+1)K_2)$ that $K = G \sqcup (r+1)K_2$. Obviously, K has 2 vertices, one edge and one bipartite component more than G_1 . So, we must have $K = G_1 \sqcup K_2$. Now, the inductive hypothesis holds the proof. \square

Lemma 11 and Theorem 4 imply the following corollary.

Corollary 2. *For a k -regular graph G of order n , $(G \nabla K_1) \sqcup rK_2$ is DQS if either of the following conditions holds:*

- (i) $k \in \{1, n - 2\}$,
- (ii) $k = 2$ and $n \geq 11$,
- (iii) $k = n - 3$ and \overline{G} has no triangles.

Lemma 12 and Theorem 4 imply the following corollary.

Corollary 3. *Let G be a k -regular graph of order n . Then $(G \nabla K_2) \sqcup rK_2$ is DQS for $k \in \{1, n - 2\}$. For $k = n - 3$, $(G \nabla K_2) \sqcup rK_2$ is DQS if \overline{G} has no triangles.*

Lemma 13 and Theorem 4 imply the following corollary.

Corollary 4. *Let G be a non-bipartite graph obtained from K_n by deleting a matching. Then $G \sqcup rK_2$ is DQS.*

Remark 5. *Some 3-regular DAS graphs are given in [25] and references therein. We can obtain DQS graphs with independent edges from Corollary 4.*

Lemmas 9 and 10 and Theorem 4 imply the following corollary.

Corollary 5. *Let G be a k -regular connected non-bipartite graph of order n . Then $G \sqcup rK_2$ is DQS if either of the following holds*

- (i) $k \in \{2, n - 1, n - 2, n - 3\}$.
- (ii) $k = n - 4$ and G is DAS.

Lemma 14 and Theorem 4 imply the following corollary.

Corollary 6. *Let G be any of the following graphs. Then $G \sqcup rK_2$ is DQS.*

- (i) *The graphs C_n (n is odd), K_n ($n \geq 4$).*
- (ii) *The graphs \overline{P}_n ($n \geq 5$).*
- (iii) *The wheel graph $K_1 \nabla C_n$.*
- (iv) *Every lollipop graph $H_{n,p}$ when p is odd and $n \geq 8$.*
- (v) *The kite graph $Ki_{n,n-1}$ for $n \geq 4$ and $n \neq 5$.*
- (vi) *The friendship graph F_n .*
- (vii) *$(C_n \circ tK_1)$, when n is odd and $n \notin \{32, 64\}$ and $t \in \{1, 2\}$.*
- (viii) *$U_{r,n-r}$ if $r(\geq 3)$ is odd and $n \geq 7$.*
- (ix) *$CS(n, \alpha)$ when $1 \leq \alpha \leq n - 1$ and $\alpha \neq 3$.*
- (x) *$S(n, c)$ and its complement where $n \geq 2c + 1$ and $c \geq 1$.*
- (xi) *$H \nabla K_m$ where H is an $(n - 2)$ -regular graph on n vertices, and $\overline{K}_n \nabla K_2$ except for $n = 3$.*
- (xii) *The dumbbell graphs $D_{p,k,q}$ (p or q is odd) different from $D_{3q,0,q}$ and all non-bipartite theta graphs $\Theta_{r,s,t}$.*

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Нехай G простий ненапрявлений граф. Тоді беззнакова лапласіанова матриця G визначається як $D_G + A_G$, де D_G і A_G позначають матрицю степенів і матрицю суміжності графу G відповідно. Граф G називають визначеним своїм беззнаковим лапласіановим спектром (скорочення DQS), якщо будь-який граф, що має такий самий беззнаковий лапласіановий спектр як G , є ізоморфним до G . У роботі показано, що $G \sqcup rK_2$ визначений своїм беззнаковим лапласіановим спектром за певних умов, де r і K_2 позначають натуральне число і повний граф на двох вершинах відповідно. Застосовуючи ці результати ми отримали деякі DQS графи з незалежними вершинами.

Ключові слова і фрази: спектральна характеристика, беззнаковий лапласіановий спектр, ко-спектральні графи, об'єднання графів.



STOROZH O.G.

SOME ANALYTIC PROPERTIES OF THE WEYL FUNCTION OF A CLOSED LINEAR RELATION

Let L and L_0 , where L is an expansion of L_0 , be closed linear relations (multivalued operators) in a Hilbert space H . In terms of abstract boundary operators (i.e. in the form which in the case of differential operators leads immediately to boundary conditions) some analytic properties of the Weyl function $M(\lambda)$ corresponding to a certain boundary pair of the couple (L, L_0) are studied.

In particular, applying Hilbert resolvent identity for relations, the criterion of invertibility in the algebra of bounded linear operators in H for transformation $M(\lambda) - M(\lambda_0)$ in certain small punctured neighbourhood of λ_0 is established. It is proved that in this case λ_0 is a first-order pole for the operator-function $(M(\lambda) - M(\lambda_0))^{-1}$. The corresponding residue and Laurent series expansion are found.

Under some additional assumptions, the behaviour of so called γ -field Z_λ (being an operator-function closely connected to $M(\lambda)$) as $\lambda \rightarrow -\infty$ is investigated.

Key words and phrases: Hilbert space, relation, operator, extension, pole.

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INTRODUCTION

The theory of linear relations (multivalued operators) in Hilbert space was initiated by R. Arens [1]. Various aspects of the extension theory of linear relations (in particular, nondensely defined operators; first of all, Hermitian ones) were studied by a number of authors (see, e.g. [3, 15, 16], [5]– [8], [9], [10], [14]).

Let us explain that under (closed) linear relation in H , where H is a fixed complex Hilbert space equipped with inner product $(\cdot|\cdot)$, we understand a (closed) linear manifold in $H^2 \stackrel{def}{=} H \oplus H$ and that in the theory of linear relations every linear operator is identified with its graph. Each such relation T has the adjoint T^* which is defined as follows:

$$T^* = H^2 \ominus JT \left(= J(H^2 \ominus T) \right)$$

(here and below \oplus and \ominus are the symbols of orthogonal sum and orthogonal complement, respectively; for all $h_1, h_2 \in H$ $J(h_1, h_2) \stackrel{def}{=} (-ih_2, ih_1)$).

In this paper the role of initial object is played by two couples (L, L_0) and (M, M_0) of closed linear relations in H such that

$$L_0 \subset L, \quad M = L_0^*, \quad M_0 = L^*.$$

Let us note that in [13, 15–17] the term "dual pair" was using instead of "couple" in the present paper. The authors of [2] were using the term *adjoint pair*.

The notion of Weyl function had been introduced at first in [4] under the assumption that L_0 is a nonnegative densely defined operator and $M = L$. Later on it was extended onto more wide varieties of operators and relations in some of papers, mentioned above (e.g. [5, 7, 15, 16]). It turned out that this notion is very important in the extension theory, since certain classes of extensions of a given operator or relation may be described by using this notion.

In this article (which can be regarded as a continuation of investigations originated in [20, 21]) we study some analytic properties of the Weyl function of (L, L_0) corresponding to the certain its boundary pair (see Definitions 1, 2).

1 NOTATIONS AND PRELIMINARY RESULTS

Through this paper we use the following notations:

$D(T)$, $R(T)$, $\ker T$ are, respectively, the domain, range, and kernel of a (linear) relation (in partial, operator) T :

$$D(T) = \{y \in H \mid (\exists y' \in H) : (y, y') \in T\}; \quad R(T) = \{y' \in H \mid (\exists y \in H) : (y, y') \in T\};$$

$$\ker T = \{y \in H \mid (y, 0) \in T\};$$

if $\lambda \in \mathbb{C}$ then $T - \lambda = \{(y, y' - \lambda y) \mid (y, y') \in T\}$, sequently

$$\ker(T - \lambda) = \{y \in H \mid (y, 0) \in T - \lambda\} (= \{y \in H \mid (y, \lambda y) \in T\});$$

$$T^{-1} = \{(y', y) \in H^2 \mid (y, y') \in T\};$$

$$\rho(T) = \{\lambda \in \mathbb{C} \mid \ker(T - \lambda) = \{0\}, R(T - \lambda) = H\} \text{ (the resolvent set of } T\text{);}$$

1_X is the identity in X ;

$+$, $\dot{+}$ are, respectively, the symbols of sum and direct sum in a linear space.

If X, Y are Hilbert spaces then $(\cdot | \cdot)_X$ is the symbol of scalar product in X , $\mathcal{B}(X, Y)$ is the set of linear bounded operators $A : X \rightarrow Y$ such that $D(A) = X$; $\mathcal{B}(X) \stackrel{\text{def}}{=} \mathcal{B}(X, X)$.

If $A_i : X \rightarrow Y_i$ ($i = 1, 2$) are linear operators then the notation $A = A_1 \oplus A_2$ means that $Ax = \begin{pmatrix} A_1x \\ A_2x \end{pmatrix}$ for every $x \in X$.

Definition 1 ([18]). *Let G be an (auxiliary) Hilbert space and $\Gamma \in \mathcal{B}(L, G)$. The pair (G, Γ) is called a boundary pair for (L, L_0) if $R(\Gamma) = G$, $\ker \Gamma = L_0$.*

Theorem 1 ([18, 19]). *There exist Hilbert spaces G_1, G_2 and the operators*

$$\Gamma_1 \in \mathcal{B}(L, G_1), \Gamma_2 \in \mathcal{B}(L, G_2), \tilde{\Gamma}_1 \in \mathcal{B}(M, G_2), \tilde{\Gamma}_2 \in \mathcal{B}(M, G_1)$$

such that

i) $(G_1 \oplus G_2, \Gamma_1 \oplus \Gamma_2)$ is a boundary pair for (L, L_0) ;

ii) $(G_2 \oplus G_1, \tilde{\Gamma}_1 \oplus \tilde{\Gamma}_2)$ is a boundary pair for (M, M_0) ;

iii) for all $\hat{y} = (y, y') \in L$, for all $\hat{z} = (z, z') \in M$ $(y'|z) - (y|z') = (\Gamma_1 \hat{y} | \tilde{\Gamma}_2 \hat{z})_{G_1} - (\Gamma_2 \hat{y} | \tilde{\Gamma}_1 \hat{z})_{G_2}$.

We suppose below that the resolvent set $\rho(L_2)$ of the relation $L_2 \stackrel{\text{def}}{=} \ker \Gamma_2$ is not empty and $\lambda \in \rho(L_2)$. Then $\bar{\lambda} \in \rho(M_2)$, where $M_2 \stackrel{\text{def}}{=} \ker \tilde{\Gamma}_2 (= L_2^*)$ and

$$L_\lambda \stackrel{def}{=} (L_2 - \lambda)^{-1} \in \mathcal{B}(H), \quad M_{\bar{\lambda}} \stackrel{def}{=} (M_2 - \bar{\lambda})^{-1} (= L_\lambda^*) \in \mathcal{B}(H).$$

$$\text{Put for all } y \in H \quad \hat{L}_\lambda y = \begin{pmatrix} L_\lambda y \\ y + \lambda L_\lambda y \end{pmatrix}, \text{ for all } z \in H \quad \hat{M}_{\bar{\lambda}} z = \begin{pmatrix} M_{\bar{\lambda}} z \\ z + \bar{\lambda} M_{\bar{\lambda}} z \end{pmatrix},$$

$$\text{for all } \hat{y} = (y, y') \in H^2 \quad \tilde{L}_\lambda \hat{y} = L_\lambda y + (y' + \lambda L_\lambda y'),$$

$$\text{for all } \hat{z} = (z, z') \in H^2 \quad \tilde{M}_{\bar{\lambda}} \hat{z} = M_{\bar{\lambda}} z + (z' + \bar{\lambda} M_{\bar{\lambda}} z')$$

(it is clear that $\hat{L}_\lambda^* = \tilde{M}_{\bar{\lambda}}$, $\hat{M}_{\bar{\lambda}}^* = \tilde{L}_\lambda$),

$$Z_\lambda = (\tilde{\Gamma}_1 \hat{M}_{\bar{\lambda}})^* (= \tilde{L}_\lambda \tilde{\Gamma}_1^*), \quad \tilde{Z}_{\bar{\lambda}} = (\Gamma_1 \hat{L}_\lambda)^* (= \tilde{M}_{\bar{\lambda}} \Gamma_1^*), \quad \hat{Z}_\lambda = \begin{pmatrix} Z_\lambda \\ \lambda Z_\lambda \end{pmatrix}.$$

Note that in some articles Z_λ is said to be a γ -field.

Lemma 1 ([19]). *i) $R(\hat{L}_\lambda) = L_2$, $R(\hat{M}_{\bar{\lambda}}) = M_2$;*

ii) $Z_\lambda \in \mathcal{B}(G_2, H)$ and $R(Z_\lambda) = \ker(L - \lambda)$,

$\tilde{Z}_{\bar{\lambda}} \in \mathcal{B}(G_1, H)$ and $R(\tilde{Z}_{\bar{\lambda}}) = \ker(M - \bar{\lambda})$;

iii) $R(\hat{Z}_\lambda) \subset L$ and $\Gamma_2 \hat{Z}_\lambda = 1_{G_2}$.

Proposition 1 ([1,3,6]). *Let S be closed linear nonnegative, in symbols $S \geq 0$ (that is $(z|y) \geq 0$ for all $(y, z) \in S$), selfadjoint relation in H and $\lambda < 0$. Then*

i)

$$\lambda \in \rho(S), \quad \left\| (S - \lambda)^{-1} \right\| \leq \frac{1}{|\lambda|}. \quad (1)$$

ii) Put $S(0) = \{y \in H : (0, y) \in S\}$, $S_s = S \ominus (\{0\} \oplus S(0))$. S_s is the graph of selfadjoint operator $S_{op} : S(0)^\perp \rightarrow S(0)^\perp (\equiv \overline{D(S)})$ (which is said to be an operator part of S) with $D(S_{op}) = D(S)$ and $R((S - \lambda)_{op}) = S(0)^\perp$.

It is clear that (1) implies

$$\text{for all } f \in H \quad \lim_{\lambda \rightarrow -\infty} (S - \lambda)^{-1} f = 0. \quad (2)$$

Moreover, if S is an operator, then

$$\text{for all } f \in H \quad \lim_{\lambda \rightarrow -\infty} [\lambda (S - \lambda)^{-1} f + f] = 0. \quad (3)$$

Indeed, for each $g \in D(S)$ we have $\lambda (S - \lambda)^{-1} g + g = (S - \lambda)^{-1} Sg \xrightarrow{\lambda \rightarrow -\infty} 0$ (see (2)). Further, in view of (1) for all $\lambda \in (-\infty, 0)$ $\left\| \lambda (S - \lambda)^{-1} + 1_H \right\| \leq 2$.

Since $\overline{D(S)} = H$, two latter relations guarantee that (3) is true. It follows from the well known criterion of the strong convergence for the operator sequences (see [12, p. 59]).

2 AUXILIARY STATEMENTS

Remark 1. *Applying Hilbert resolvent identity for relations (see [6]) it is easy to prove that*

$$\text{for all } \lambda, \mu \in \rho(L_2) \quad \tilde{L}_\lambda - \tilde{L}_\mu = (\lambda - \mu) L_\lambda \tilde{L}_\mu (= (\lambda - \mu) L_\mu \tilde{L}_\lambda). \quad (4)$$

Indeed,

$$\begin{aligned}\tilde{L}_\lambda - \tilde{L}_\mu &= (L_\lambda - L_\mu, \lambda L_\lambda - \mu L_\mu) = (L_\lambda - L_\mu, (\lambda - \mu) L_\lambda + \mu (L_\lambda - L_\mu)) \\ &= ((\lambda - \mu) L_\lambda L_\mu, (\lambda - \mu) L_\lambda + \mu (\lambda - \mu) L_\lambda L_\mu) \\ &= (\lambda - \mu) L_\lambda (L_\mu, 1_H + \mu L_\mu) = (\lambda - \mu) L_\lambda \tilde{L}_\mu.\end{aligned}$$

Similar arguments show that

$$\text{for all } \lambda, \mu \in \rho(L_2) \quad \hat{L}_\lambda - \hat{L}_\mu = (\lambda - \mu) \hat{L}_\lambda L_\mu (= (\lambda - \mu) \hat{L}_\mu L_\lambda).$$

Lemma 2. *Let $\lambda, \mu \in \rho(L_2)$. Then*

$$Z_\lambda - Z_\mu = (\lambda - \mu) L_\lambda Z_\mu (= (\lambda - \mu) L_\mu Z_\lambda), \quad (5)$$

$$\tilde{Z}_\lambda^* - \tilde{Z}_\mu^* = (\lambda - \mu) \tilde{Z}_\mu^* L_\lambda (= (\lambda - \mu) \tilde{Z}_\lambda^* L_\mu), \quad (6)$$

$$\hat{Z}_\lambda - \hat{Z}_\mu = (\lambda - \mu) \hat{L}_\lambda Z_\mu (= (\lambda - \mu) \hat{L}_\mu Z_\lambda). \quad (7)$$

Proof. Taking into account (4) we obtain

$$Z_\lambda - Z_\mu = (\tilde{L}_\lambda - \tilde{L}_\mu) \tilde{\Gamma}_1^* = (\lambda - \mu) L_\lambda \tilde{L}_\mu \tilde{\Gamma}_1^* = (\lambda - \mu) L_\lambda Z_\mu.$$

The equality (5) is proved. The proof of (6) is analogous. Furthermore,

$$\begin{aligned}\lambda Z_\lambda - \mu Z_\mu &= \lambda (Z_\lambda - Z_\mu) + (\lambda - \mu) Z_\mu \\ &= \lambda (\lambda - \mu) L_\lambda Z_\mu + (\lambda - \mu) Z_\mu = (\lambda - \mu) (1_H + \lambda L_\lambda) Z_\mu.\end{aligned}$$

The latter identity together (5) implies (7). □

Corollary 1. *For arbitrary $\lambda \in \rho(L_2)$, $n \in \mathbb{N}$ we have*

$$Z_\lambda^{(n)} = n! L_\lambda^n Z_\lambda, \quad (8)$$

$$(\tilde{Z}_\lambda^*)^{(n)} = n! \tilde{Z}_\lambda^* L_\lambda^n, \quad (9)$$

$$\hat{Z}_\lambda^{(n)} = n! \hat{L}_\lambda L_\lambda^{n-1} Z_\lambda. \quad (10)$$

Proof. First of all, note that

$$L_\lambda^{(n)} = n! L_\lambda^{n+1}, \quad \tilde{L}_\lambda^{(n)} = n! L_\lambda^n \tilde{L}_\lambda, \quad \hat{L}_\lambda^{(n)} = n! \hat{L}_\lambda L_\lambda^n. \quad (11)$$

In the case $n = 1$ these equalities follow immediately from the Hilbert resolvent identity. In the general case induction should be applied. The equalities (11) imply (8), (9). In order to prove (10) note that $(\lambda Z_\lambda)^{(n)} = n Z_\lambda^{(n-1)} + \lambda Z_\lambda^{(n)}$ (it can be shown by induction). The latter identity together with (9) imply (11). □

Lemma 3. *Suppose that $\lambda, \mu \in \rho(L_2)$. Then*

$$\left(\tilde{Z}_\mu^* Z_\lambda \right)^{-1} \in \mathcal{B}(G_1, G_2) \Leftrightarrow R(L_0 - \mu) + \ker(L - \lambda) = H. \quad (12)$$

Proof. It is sufficient to verify the next implications:

- i) $R(L_0 - \mu) \cap \ker(L - \lambda) = \{0\} \Rightarrow \ker(\tilde{Z}_\mu^* Z_\lambda) = \{0\}$,
- ii) $R(L_0 - \mu) + \ker(L - \lambda) = H \Rightarrow R(\tilde{Z}_\mu^* Z_\lambda) = G_1$,
- iii) $\ker(\tilde{Z}_\mu^* Z_\lambda) = \{0\} \Rightarrow R(L_0 - \mu) \cap \ker(L - \lambda) = \{0\}$,
- iv) $R(\tilde{Z}_\mu^* Z_\lambda) = G_1 \Rightarrow R(L_0 - \mu) + \ker(L - \lambda) = H$.

Let us consider each of them.

i) Assume that for some $a \in G_2$ the equality $\tilde{Z}_\mu^* Z_\lambda a \equiv \Gamma_1 \hat{L}_\mu Z_\lambda a = 0$ holds. Then $\hat{L}_\mu Z_\lambda a \in \ker \Gamma_1$. But $\hat{L}_\mu Z_\lambda a \in L_2 = \ker \Gamma_2$, hence $\hat{L}_\mu Z_\lambda a \in L_0$. In other words, $\begin{pmatrix} L_\mu Z_\lambda a \\ Z_\lambda a + \mu L_\mu Z_\lambda a \end{pmatrix} \in L_0$. Consequently $\begin{pmatrix} L_\mu Z_\lambda a \\ Z_\lambda a \end{pmatrix} \in L_0 - \mu$, in particular $Z_\lambda a \in R(L_0 - \mu)$. But (see Lemma 1) $Z_\lambda a \in \ker(L - \lambda)$, therefore $Z_\lambda a = 0$, $\hat{Z}_\lambda a = 0$. Thus $a = \Gamma_2 \hat{Z}_\lambda a = 0$.

ii) For arbitrary $h \in G_1$ there exists $\hat{y} = \begin{pmatrix} y \\ y' \end{pmatrix} \in L_2 = \ker \Gamma_2$ satisfying the equality $\Gamma_1 \hat{y} = h$. We have: $\begin{pmatrix} y \\ y' - \mu y \end{pmatrix} \in L_2 - \mu$, in particular, $y = L_\mu(y' - \mu y)$. Further, there exist $u \in R(L_0 - \mu)$, $a \in G_2$ such that $y' - \mu y = u + Z_\lambda a$. It means that for some $\hat{y}_0 = \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix} \in H^2$ the equalities $u = y'_0 - \mu y_0$, $y_0 = L_\mu(y'_0 - \mu y_0)$ are fulfilled. Whence using these equalities we obtain

$$\begin{aligned} \hat{L}_\mu Z_\lambda a &= \hat{L}_\mu(y' - \mu y - u) = \hat{L}_\mu((y' - \mu y) - (y'_0 - \mu y_0)) = \hat{L}_\mu(y' - \mu y) - \hat{L}_\mu(y'_0 - \mu y_0) \\ &= \begin{pmatrix} L_\mu(y' - \mu y) \\ y' - \mu y + \mu L_\mu(y' - \mu y) \end{pmatrix} - \begin{pmatrix} L_\mu(y'_0 - \mu y_0) \\ y'_0 - \mu y_0 + \mu L_\mu(y'_0 - \mu y_0) \end{pmatrix} \\ &= \begin{pmatrix} y \\ y' - \mu y + \mu y \end{pmatrix} - \begin{pmatrix} y_0 \\ y'_0 - \mu y_0 + \mu y_0 \end{pmatrix} = \begin{pmatrix} y \\ y' \end{pmatrix} - \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix} = \hat{y} - \hat{y}_0, \end{aligned}$$

consequently

$$\tilde{Z}_\mu^* Z_\lambda a = \Gamma_1 \hat{L}_\mu Z_\lambda a = \Gamma_1(\hat{y} - \hat{y}_0) = \Gamma_1 \hat{y} = h.$$

iii) Assume that $y \in R(L_0 - \mu) \cap \ker(L - \lambda)$. Then $y = Z_\lambda a$ for some $a \in G_2$ and

$$y \in R(L_0 - \mu). \quad (13)$$

The inclusion (13) implies $y \in R(L_2 - \mu)$. It is easy to see that

$$(L_\mu y, y) \in L_2 - \mu. \quad (14)$$

Taking into account (13), (14) and the equality $\ker(L_2 - \lambda) = \{0\}$, we obtain $(L_\mu y, y) \in L_0 - \mu$. The latter inclusion yields $\hat{L}_\mu y = \begin{pmatrix} L_\mu y \\ y + \mu L_\mu y \end{pmatrix} \in L_0$, therefore $\Gamma_1 \hat{L}_\mu y = \Gamma_1 \hat{L}_\mu Z_\lambda a = 0$.

Now it is clear that $a = 0$, $y = 0$.

iv) For any $h \in H$ we have $\hat{L}_\mu h \in L_2$ (see Lemma 1). Put $\Gamma_1 \hat{L}_\mu h = g$. There exists $u \in \ker(L - \lambda) = R(Z_\lambda)$ such that $\Gamma_1 \hat{L}_\mu u = g$, consequently $\Gamma_1 \hat{L}_\mu(h - u) = 0$. Moreover, $\hat{L}_\mu(h - u) \in L_0$, i.e. $\begin{pmatrix} L_\mu(h - u) \\ h - u \end{pmatrix} \in L_0 - \mu$. Thus $h = u + (h - u) \in \ker(L - \lambda) + R(L_0 - \mu)$. \square

Remark 2. Assume that $\lambda_0 \in \rho(L_2)$, $(\tilde{Z}_{\lambda_0}^* Z_{\lambda_0})^{-1} \in \mathcal{B}(G_1, G_2)$, and

$$0 < |\lambda - \lambda_0| < \min \left\{ \frac{1}{2 \|L_{\lambda_0}\|}, \frac{1}{2 \|\tilde{Z}_{\lambda_0}^*\| \cdot \|Z_{\lambda_0}\| \cdot \|L_{\lambda_0}^{-1}\| \cdot \|\tilde{Z}_{\lambda_0}^* Z_{\lambda_0}\|} \right\}. \tag{15}$$

Then $(\tilde{Z}_{\lambda}^* Z_{\lambda})^{-1} \in \mathcal{B}(G_1, G_2)$ (here and below $\|T\|$ is the norm of operator T).

Indeed, let $|\lambda - \lambda_0| < \frac{1}{2 \|L_{\lambda_0}\|}$. Applying the theorem on perturbation of invertible in $\mathcal{B}(H)$ operator (see [11, pp. 228–229]) we see that $(1_H + (\lambda - \lambda_0) L_{\lambda_0})^{-1} \in \mathcal{B}(H)$ and

$$\left\| (1_H + (\lambda - \lambda_0) L_{\lambda_0})^{-1} - 1_H \right\| < 2 |\lambda - \lambda_0| \|L_{\lambda_0}\|. \tag{16}$$

Further, taking into account (6) in which μ is replaced by λ_0 we conclude that

$$\tilde{Z}_{\lambda}^* Z_{\lambda} - \tilde{Z}_{\lambda_0}^* Z_{\lambda_0} = \tilde{Z}_{\lambda_0}^* \left[(1_H + (\lambda - \lambda_0) L_{\lambda_0})^{-1} - 1_H \right] Z_{\lambda_0}.$$

Whence using above-mentioned theorem and (16) we obtain the following: (12) implies $(\tilde{Z}_{\lambda}^* Z_{\lambda})^{-1} \in \mathcal{B}(G_1, G_2)$.

Proposition 2. Suppose that $M_0 = L_0 \geq 0$, $M = L$, $G_1 = G_2 \stackrel{def}{=} \mathcal{H}$, $\tilde{\Gamma}_1 = \Gamma_1$, $\tilde{\Gamma}_2 = \Gamma_2$ (in other words, $(\mathcal{H}, \Gamma_1, \Gamma_2)$ is a boundary triple (boundary value space) of L [5, 9, 10, 15]). Let $L_2 \stackrel{def}{=} \ker \Gamma_2$ be a (selfadjoint) nonnegative extension of L_0 , and L_{λ} , \tilde{L}_{λ} , Z_{λ} be as above.

Under these assumptions

$$s - \lim_{\lambda \rightarrow -\infty} Z_{\lambda}^* = \Gamma_1 (0 \oplus Q), \quad w - \lim_{\lambda \rightarrow -\infty} Z_{\lambda} = Q \pi_2 \Gamma_1^*, \tag{17}$$

where $s - \lim$ and $w - \lim$ are respectively the symbols of strong and weak limits for operator-functions, while Q and π_2 are the orthoprojections $H \rightarrow L_2(0)$ and $H^2 \rightarrow \{0\} \oplus H$.

Proof. Let $f \in H$, P be the orthoprojection $H \rightarrow R((L_2 - \lambda)_{op}) (= L_2(0)^{\perp})$, and Q be the orthoprojection $H \rightarrow (L_2 - \lambda)(0) (= L_2(0))$. Then $f = Pf + Qf$. We obtain

$$L_{\lambda} f = L_{\lambda} P f + L_{\lambda} Q f = \left((L_2 - \lambda)_{op} \right)^{-1} P f.$$

But $(L_2 - \lambda)_{op} = L_{2,op} - \lambda$ (indeed,

for all $f \in D((L_2 - \lambda)_{op}) = D(L_{2,op} - \lambda) (= D(L_2))$ $(L_{2,op} - \lambda) f - (L_2 - \lambda)_{op} f \in L_2(0)^{\perp}$;

on the other hand, the inclusions $(f, (L_{2,op} - \lambda) f)$, $(f, (L_2 - \lambda)_{op} f) \in L_2 - \lambda$ imply $(L_2 - \lambda)_{op} f - (L_{2,op} - \lambda) f \in (L_2 - \lambda)(0) = L_2(0)$, therefore

$$\lambda L_{\lambda} f + f = \lambda (L_{2,op} - \lambda)^{-1} P f + P f + Q f.$$

Taking into account (3) with $S = L_{2,op}$ we see that $\lim_{\lambda \rightarrow -\infty} (\lambda L_{\lambda} f + f) = Q f$, whence using (2) with $S = L_2$ we obtain $\lim_{\lambda \rightarrow -\infty} \hat{L}_{\lambda} f = \begin{pmatrix} 0 \\ Q f \end{pmatrix}$, therefore $\lim_{\lambda \rightarrow -\infty} Z_{\lambda}^* f = \lim_{\lambda \rightarrow -\infty} \Gamma_1 \hat{L}_{\lambda} f = \Gamma_1 (0 \oplus Q) f$. The first of the equalities (17) has been proved. The second equality is an immediate consequence from the first one. \square

3 MAIN RESULT

Definition 2 ([16]). An operator-function $M(\lambda) \stackrel{\text{def}}{=} \Gamma_1 \hat{Z}_\lambda$ ($\lambda \in \rho(L_2)$) is called the Weyl function of the couple (L, L_0) corresponding to its boundary pair $(G_1 \oplus G_2, \Gamma_1 \oplus \Gamma_2)$.

Lemma 4. For any $\lambda, \mu \in \rho(L_2)$, the equality

$$M(\lambda) - M(\mu) = (\lambda - \mu) \tilde{Z}_\mu^* Z_\lambda \left(= (\lambda - \mu) \tilde{Z}_\lambda^* Z_\mu \right)$$

is true.

Proof. In view of (10) we obtain

$$M(\lambda) - M(\mu) = \Gamma_1 (\hat{Z}_\lambda - \hat{Z}_\mu) = (\lambda - \mu) \Gamma_1 \hat{L}_\lambda Z_\mu = (\lambda - \mu) \tilde{Z}_\mu^* Z_\lambda \left(= (\lambda - \mu) \tilde{Z}_\lambda^* Z_\mu \right).$$

□

Consider some analytic properties of the operator-function $M(\lambda)$.

Lemma 5. $M(\lambda)$ is analytic $\mathcal{B}(G_1, G_2)$ -valued function on $\rho(L_2)$. Moreover, for any $n \in \mathbb{N}$

$$M^{(n)}(\lambda) = n! \tilde{Z}_\lambda^* L_\lambda^{n-1} Z_\lambda, \tag{18}$$

in particular $M'(\lambda) = \tilde{Z}_\lambda^* Z_\lambda$.

Proof. Since L_λ is a $\mathcal{B}(H)$ -valued analytic function on $\rho(L_2)$, we conclude that

$$\hat{Z}_\lambda = \begin{pmatrix} L_\lambda & 1_H + \lambda L_\lambda \\ \lambda L_\lambda & \lambda 1_H + \lambda^2 L_\lambda \end{pmatrix} \tilde{\Gamma}_1^*$$

is an analytic $\mathcal{B}(G_2, H^2)$ -valued function. But by virtue of Lemma 1 $R(\hat{Z}_\lambda) \subset L$, sequently \hat{Z}_λ is a $\mathcal{B}(G_2, L)$ -valued analytic function. Moreover (see (10)) $\hat{Z}_\lambda^{(n)} = n! \hat{L}_\lambda L_\lambda^{n-1} Z_\lambda$, therefore

$$M^{(n)}(\lambda) = \Gamma_1 \hat{Z}_\lambda^{(n)} = n! \Gamma_1 \hat{L}_\lambda L_\lambda^{n-1} Z_\lambda.$$

□

Theorem 2. Suppose that $\lambda_0 \in \rho(L_2)$, $R(L_0 - \lambda_0) \dot{+} \ker(L - \lambda) = H$, and (15) holds. Then

- i) $(\tilde{Z}_{\lambda_0}^* Z_{\lambda_0})^{-1} \in \mathcal{B}(G_1, G_2)$, $(M(\lambda) - M(\lambda_0))^{-1} \in \mathcal{B}(G_1, G_2)$;
- ii) λ_0 is a first-order pole for the function $(M(\lambda) - M(\lambda_0))^{-1}$ and

$$\text{res}_{|\lambda=\lambda_0} (M(\lambda) - M(\lambda_0))^{-1} = (\tilde{Z}_{\lambda_0}^* Z_{\lambda_0})^{-1}.$$

Proof. i) This statement is a direct consequence of Lemma 3, Remark 2 and Lemma 4.

ii) Put

$$\Pi(\lambda) = \begin{cases} (\lambda - \lambda_0)^{-1} (M(\lambda) - M(\lambda_0)), & \lambda \neq \lambda_0 \\ M'(\lambda_0) = \tilde{Z}_{\lambda_0}^* Z_{\lambda_0}, & \lambda = \lambda_0 \end{cases}.$$

It is clear that $\lim_{\lambda \rightarrow \lambda_0} \Pi(\lambda) = M'(\lambda_0) = \tilde{Z}_{\lambda_0}^* Z_{\lambda_0}$ (with respect to uniform operator convergence). Hence, $\lim_{\lambda \rightarrow \lambda_0} [(\lambda - \lambda_0)(M(\lambda) - M(\lambda_0))^{-1}] = \lim_{\lambda \rightarrow \lambda_0} \Pi(\lambda)^{-1} = (\tilde{Z}_{\lambda_0}^* Z_{\lambda_0})^{-1}$. The theorem is proved. □

Remark 3. *Theorem 2 yields that in some neighbourhood of the point $\lambda_0 \in \rho(L_2)$ such that $R(L_0 - \lambda_0) + \ker(L - \lambda) = H$, the following expansion takes place:*

$$(M(\lambda) - M(\lambda_0))^{-1} = \frac{1}{\lambda - \lambda_0} \left(\tilde{Z}_{\lambda_0}^* Z_{\lambda_0} \right)^{-1} + \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n R^{(n)}, \quad (19)$$

where $R^{(n)} \in \mathcal{B}(G_1, G_2)$, $n = 0, 1, 2, \dots$. On the other hand, in view of (18) we obtain

$$M(\lambda) - M(\lambda_0) = \sum_{n=1}^{\infty} (\lambda - \lambda_0)^n \cdot \tilde{Z}_{\lambda_0}^* L_{\lambda_0}^{n-1} Z_{\lambda_0}.$$

Multiplying both sides of two latter equalities we obtain the recurrent relations for the coefficients $R^{(n)}$ in (19):

$$\sum_{m=0}^n \tilde{Z}_{\lambda_0}^* L_{\lambda_0}^m Z_{\lambda_0} \cdot R^{(n-m-1)} = 0 \quad (n \in \mathbb{N}), \quad R^{(-1)} = (\tilde{Z}_{\lambda_0}^* Z_{\lambda_0})^{-1}.$$

In particular, $R^{(0)} = -R^{(-1)} \cdot \tilde{Z}_{\lambda_0}^* L_{\lambda_0} Z_{\lambda_0} \cdot R^{(-1)}$.

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Нехай L та L_0 , де $L_0 \subset L$, — замкнені лінійні відношення (багатозначні оператори) у комплексному гільбертовому просторі H . У термінах абстрактних граничних операторів (тобто у вигляді, який у випадку диференціальних операторів приводить безпосередньо до граничних умов) досліджуються деякі аналітичні властивості функції Вейля $M(\lambda)$, яка відповідає деякій граничній парі (L, L_0) .

Зокрема, застосовуючи резольвентну тотожність Гільберта для відношень, встановлено критерій оборотності у алгебрі обмежених лінійних операторів, діючих у H , для відображення $M(\lambda) - M(\lambda_0)$ у деякому достатньо малому проколеному околі точки λ_0 . Доведено, що в цьому випадку λ_0 є полюсом першого порядку для оператор-функції $(M(\lambda) - M(\lambda_0))^{-1}$. Знайдено відповідні лишок та розвинення у ряд Лорана.

При деяких додаткових припущеннях досліджується поведінка при $\lambda \rightarrow -\infty$ так званого γ -поля Z_λ , яке являє собою оператор-функцію, тісно пов'язану з $M(\lambda)$.

Ключові слова і фрази: гільбертів простір, відношення, оператор, розширення, полюс.



VASYLYSHYN T.V.

SOME PROPERTIES OF SHIFT OPERATORS ON ALGEBRAS GENERATED BY *-POLYNOMIALS

A $*$ -polynomial is a function on a complex Banach space X , which is a sum of so-called (p, q) -polynomials. In turn, for non-negative integers p and q , a (p, q) -polynomial is a function on X , which is the restriction to the diagonal of some mapping, defined on the Cartesian power X^{p+q} , which is linear with respect to every of its first p arguments and antilinear with respect to every of its other q arguments. The set of all continuous $*$ -polynomials on X form an algebra, which contains the algebra of all continuous polynomials on X as a proper subalgebra. So, completions of this algebra with respect to some natural norms are wider classes of functions than algebras of holomorphic functions. On the other hand, due to the similarity of structures of $*$ -polynomials and polynomials, for the investigation of such completions one can use the technique, developed for the investigation of holomorphic functions on Banach spaces.

We investigate the Fréchet algebra of functions on a complex Banach space, which is the completion of the algebra of all continuous $*$ -polynomials with respect to the countable system of norms, equivalent to norms of the uniform convergence on closed balls of the space. We establish some properties of shift operators (which act as the addition of some fixed element of the underlying space to the argument of a function) on this algebra. In particular, we show that shift operators are well-defined continuous linear operators. Also we prove some estimates for norms of values of shift operators. Using these results, we investigate one special class of functions from the algebra, which is important in the description of the spectrum (the set of all maximal ideals) of the algebra.

Key words and phrases: (p, q) -polynomial, $*$ -polynomial, shift operator.

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INTRODUCTION

$*$ -Polynomials (see definition below) are natural generalizations of usual polynomials on complex Banach spaces. Such objects were firstly studied in [4]. It is known that completions of the algebra of all continuous polynomials on some complex Banach space with respect to topologies of uniform convergence on some bounded subsets of the space are algebras of holomorphic functions. On the other hand, the analogical completions of the algebra of all continuous $*$ -polynomials contain wider classes of continuous functions. Except of holomorphic functions, they can contain functions, which are complex-conjugate to holomorphic. Also, as it is shown in [3], such algebras can contain functions, which cannot be represented as linear combination of products of holomorphic functions and complex-conjugate to holomorphic functions. Thus, such algebras can contain the wide enough class of continuous functions on a complex Banach space. The algebraic structure gives the opportunity to consider the elements

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of the algebra as continuous functions on the spectrum (the set of maximal ideals) of the algebra. In the description of spectra of algebras of functions on Banach spaces the so-called convolution operation on the spectrum plays an important role. In turn, the convolution operation on the spectrum is defined with aid of the so-called shift operators, defined on the algebra. Shift operators for algebras of holomorphic functions on Banach spaces and their applications for the spectra were investigated in [1], [2], [7], [8].

In this work we establish some properties of shift operators on the Fréchet algebra of functions on a complex Banach space, which is the completion of the algebra of all continuous *-polynomials with respect to the countable set of norms, which are equivalent to norms of the uniform convergence on closed balls with rational radii, centered at 0. We show that shift operators are well-defined continuous linear operators. Also we investigate one special class of functions from the algebra, constructed by using of the composition of continuous linear functionals with shift operators. Such classes of functions play an important role in the description of spectra of algebras of functions on Banach spaces.

Let \mathbb{N} be the set of all positive integers and \mathbb{Q}_+ be the set of all positive rationals. Let X be a complex Banach space. A mapping $A : X^{p+q} \rightarrow \mathbb{C}$, where $p, q \in \mathbb{N} \cup \{0\}$ are such that $p \neq 0$ or $q \neq 0$, is called a (p, q) -linear mapping, if A is linear with respect to every of first p arguments and it is antilinear with respect to every of last q arguments. A (p, q) -linear mapping, which is invariant with respect to permutations of its first p arguments and last q arguments separately, is called symmetric. A mapping $P : X \rightarrow \mathbb{C}$ is called a (p, q) -polynomial if there exists a symmetric (p, q) -linear mapping $A_p : X^{p+q} \rightarrow \mathbb{C}$ such that P is the restriction to the diagonal of A_p , i.e.,

$$P(x) = A_p(\underbrace{x, \dots, x}_{p+q})$$

for every $x \in X$. The mapping A_p is called the symmetric (p, q) -linear mapping, associated with P . (p, q) -polynomials and (p, q) -linear mappings were studied in [5] and [6].

Note that for (p, q) -polynomials the following analog of the Binomial formula holds:

$$P(x + y) = \sum_{j=0}^p \sum_{k=0}^q \frac{p!q!}{j!(p-j)!k!(q-k)!} A_p(x^j, y^{p-j}, x^k, y^{q-k}), \quad (1)$$

where

$$A_p(x^j, y^{p-j}, x^k, y^{q-k}) = A_p(\underbrace{x, \dots, x}_j, \underbrace{y, \dots, y}_{p-j}, \underbrace{x, \dots, x}_k, \underbrace{y, \dots, y}_{q-k})$$

for every $x, y \in X$. Let us denote by $\mathcal{P}^{(p,q)}X$ the space of all continuous (p, q) -polynomials with norm

$$\|P\| = \sup_{\|x\| \leq 1} |P(x)|.$$

Also, for convenience, let $\mathcal{P}^{(0,0)}X = \mathbb{C}$.

A mapping $P : X \rightarrow \mathbb{C}$ is called a *-polynomial if it can be represented in the form

$$P = \sum_{p=0}^M \sum_{q=0}^N P_{pq},$$

where $M, N \in \mathbb{N} \cup \{0\}$ and $P_{pq} \in \mathcal{P}^{(p,q)}X$. Denote $\mathcal{P}_*(X)$ the algebra of all continuous *-polynomials on the space X .

1 THE MAIN RESULT

Let

$$\{\|\cdot\|_r : r \in (0, +\infty)\} \quad (2)$$

be the set of norms on $\mathcal{P}_*(X)$ such that

1. $\|PQ\|_r \leq \|P\|_r \|Q\|_r$ for every $P, Q \in \mathcal{P}_*(X)$ and $r \in (0, +\infty)$.
2. There exist functions $(0, +\infty) \ni t \mapsto c_t \in (0, +\infty)$ and $(0, +\infty) \ni t \mapsto C_t \in (0, +\infty)$ such that $\inf_{t \in [a, b]} c_t > 0$ and $\sup_{t \in [a, b]} C_t < +\infty$ for every $b > a > 0$, and

$$c_r \sup_{\|x\| \leq r} |P(x)| \leq \|P\|_r \leq C_r \sup_{\|x\| \leq r} |P(x)|$$

for every $r \in (0, +\infty)$ and $P \in \mathcal{P}_*(X)$.

Let

$$\{\|\cdot\|_r : r \in \mathbb{Q}_+\} \quad (3)$$

be the subset of the set of norms (2). Note that the set (3) is countable. Let $\mathcal{A}(X)$ be the completion of $\mathcal{P}_*(X)$ with respect to the metric, generated by the set of norms (3). It can be checked that $\mathcal{A}(X)$ is a Fréchet algebra of functions on X . By the continuity of norms from (3),

$$c_r \sup_{\|x\| \leq r} |f(x)| \leq \|f\|_r \leq C_r \sup_{\|x\| \leq r} |f(x)| \quad (4)$$

for every $r \in \mathbb{Q}_+$ and $f \in \mathcal{A}(X)$.

Theorem 1. (i). For every $x \in X$ the operator $T_x : \mathcal{A}(X) \rightarrow \mathcal{A}(X)$, defined by

$$(T_x f)(y) = f(x + y),$$

where $f \in \mathcal{A}(X)$ and $y \in X$, is a well-defined continuous linear operator such that

$$\|T_x f\|_r \leq C_r c_{r+\|x\|}^{-1} \|f\|_{r+\|x\|},$$

for every $f \in \mathcal{A}(X)$ and $r \in \mathbb{Q}_+$.

(ii). For every $f \in \mathcal{A}(X)$ and for every continuous linear functional $\varphi : \mathcal{A}(X) \rightarrow \mathbb{C}$, the function $h_{\varphi, f} : X \rightarrow \mathbb{C}$, defined by

$$h_{\varphi, f}(x) = \varphi(T_x f),$$

belongs to $\mathcal{A}(X)$, and

$$|h_{\varphi, f}(x)| \leq K C_s c_{s+\|x\|}^{-1} \|f\|_{s+\|x\|},$$

for every $x \in X$ and for every $s \in \mathbb{Q}_+$ such that φ is continuous with respect to $\|\cdot\|_s$, where $K = \sup_{\|f\|_s \leq 1} |\varphi(f)|$.

Proof. (i). Let $x \in X$. For every $f \in \mathcal{A}(X)$, since $(T_x f)(y) = f(x + y)$ and f is well-defined at $x + y$, it follows that $T_x f$ is well-defined at y . Also note that for every $r \in \mathbb{Q}_+$

$$\sup_{\|y\| \leq r} |f(x + y)| \leq \sup_{\|z\| \leq r+\|x\|} |f(z)| \leq c_{r+\|x\|}^{-1} \|f\|_{r+\|x\|},$$

i.e.,

$$\sup_{\|y\| \leq r} |(T_x f)(y)| \leq c_{r+\|x\|}^{-1} \|f\|_{r+\|x\|}. \tag{5}$$

Let $P \in \mathcal{P}_*(X)$. Let us show that $T_x P \in \mathcal{P}_*(X)$. Let $P = \sum_{p=0}^M \sum_{q=0}^N P_{pq}$, where $P_{pq} \in \mathcal{P}(^{pq}X)$. By (1),

$$(T_x P)(y) = \sum_{p=0}^M \sum_{q=0}^N \sum_{j=0}^p \sum_{k=0}^q \frac{p!q!}{j!(p-j)!k!(q-k)!} A_{P_{pq}}(x^j, y^{p-j}, x^k, y^{q-k}),$$

where $A_{P_{pq}}$ is the symmetric (p, q) -linear mapping, associated with the (p, q) -polynomial P_{pq} for every $p \in \{0, \dots, M\}$ and $q \in \{0, \dots, N\}$. Note that for fixed $x \in X$ the function $A_{P_{pq}}(x^j, y^{p-j}, x^k, y^{q-k})$ is a continuous $(p-j, q-k)$ -polynomial with respect to y . Therefore, $T_x P$ is a continuous $*$ -polynomial.

Let $f \in \mathcal{A}(X)$. Let us show that $T_x f \in \mathcal{A}(X)$. Since $\mathcal{P}_*(X)$ is dense in $\mathcal{A}(X)$, it follows that there exists the sequence $\{f_n\}_{n=1}^\infty \subset \mathcal{P}_*(X)$, which converges to f with respect to every norm from (3). Consider the sequence $\{T_x f_n\}_{n=1}^\infty$. Since $f_n \in \mathcal{P}_*(X)$, it follows that $T_x f_n \in \mathcal{P}_*(X)$. Thus, $\{T_x f_n\}_{n=1}^\infty \subset \mathcal{P}_*(X) \subset \mathcal{A}(X)$. Let us show that the sequence $\{T_x f_n\}_{n=1}^\infty$ is fundamental in $\mathcal{A}(X)$. Let $r \in \mathbb{Q}_+$. For $m, n \in \mathbb{N}$, by (4),

$$\|T_x f_m - T_x f_n\|_r \leq C_r \sup_{\|y\| \leq r} |(T_x f_m)(y) - (T_x f_n)(y)|.$$

By (5),

$$\sup_{\|y\| \leq r} |(T_x f_m)(y) - (T_x f_n)(y)| \leq c_{r+\|x\|}^{-1} \|f_m - f_n\|_{r+\|x\|}.$$

Thus,

$$\|T_x f_m - T_x f_n\|_r \leq C_r c_{r+\|x\|}^{-1} \|f_m - f_n\|_{r+\|x\|}.$$

Since the sequence $\{f_n\}_{n=1}^\infty$ is fundamental, it follows that the sequence $\{T_x f_n\}_{n=1}^\infty$ is fundamental. Since the algebra $\mathcal{A}(X)$ is complete, it follows that there exists $g \in \mathcal{A}(X)$ such that the sequence $\{T_x f_n\}_{n=1}^\infty$ converges to g . Let $y \in X$. Let us show that $(T_x f)(y) = g(y)$. Let $\rho \in \mathbb{Q}_+$ be such that $\rho > \|y\|$. Since the sequence $\{T_x f_n\}_{n=1}^\infty$ converges to g , it follows that $\{\|T_x f_n - g\|_\rho\}_{n=1}^\infty$ converges to 0. By (4),

$$\sup_{\|z\| \leq \rho} |(T_x f_n)(z) - g(z)| \leq c_\rho^{-1} \|T_x f_n - g\|_\rho.$$

Therefore,

$$|(T_x f_n)(y) - g(y)| \leq c_\rho^{-1} \|T_x f_n - g\|_\rho.$$

Consequently, the sequence $\{(T_x f_n)(y)\}_{n=1}^\infty$ converges to $g(y)$. On the other hand, by (5),

$$\sup_{\|z\| \leq \rho} |(T_x f)(z) - (T_x f_n)(z)| \leq c_{\rho+\|x\|}^{-1} \|f - f_n\|_{\rho+\|x\|}.$$

Therefore,

$$|(T_x f)(y) - (T_x f_n)(y)| \leq c_{\rho+\|x\|}^{-1} \|f - f_n\|_{\rho+\|x\|}.$$

Since $\|f - f_n\|_{\rho+\|x\|} \rightarrow 0$ as $n \rightarrow \infty$, it follows that the sequence $\{(T_x f_n)(y)\}_{n=1}^\infty$ converges to $(T_x f)(y)$. Therefore, $(T_x f)(y) = g(y)$. Thus, $T_x f = g$ and, consequently, $T_x f \in \mathcal{A}(X)$.

By (4) and (5),

$$\|T_x f\|_r \leq C_r \sup_{\|y\| \leq r} |(T_x f)(y)| \leq C_r c_{r+\|x\|}^{-1} \|f\|_{r+\|x\|} \quad (6)$$

for every $r \in \mathbb{Q}_+$.

(ii). Let $f \in \mathcal{A}(X)$ and $\varphi \in \mathcal{A}(X)'$. Note that the function $h_{\varphi, f}(x) = \varphi(T_x f)$ is well-defined at every point $x \in X$, because $T_x f$ belongs to $\mathcal{A}(X)$ and φ is well-defined on $\mathcal{A}(X)$.

Since φ is a continuous linear functional on $\mathcal{A}(X)$, there exists $s \in \mathbb{Q}_+$ such that φ is continuous with respect to the norm $\|\cdot\|_s$. Therefore, for every $f \in \mathcal{A}(X)$,

$$|\varphi(f)| \leq K \|f\|_s, \quad (7)$$

where $K = \sup_{\|f\|_s \leq 1} |\varphi(f)|$. By (7) and (6),

$$|\varphi(T_x f)| \leq K \|T_x f\|_s \leq K C_s c_{s+\|x\|}^{-1} \|f\|_{s+\|x\|},$$

i.e.,

$$|h_{\varphi, f}(x)| \leq K C_s c_{s+\|x\|}^{-1} \|f\|_{s+\|x\|}. \quad (8)$$

Let $P = \sum_{p=0}^M \sum_{q=0}^N P_{pq}$ be a continuous $*$ -polynomial. Let us show that a function $h_{\varphi, P}(x) = \varphi(T_x P)$ is a continuous $*$ -polynomial. By (1), taking into account the linearity of φ , we have

$$h_{\varphi, P}(x) = \sum_{p=0}^M \sum_{q=0}^N \sum_{j=0}^p \sum_{k=0}^q \frac{p!q!}{j!(p-j)!k!(q-k)!} \varphi(y \mapsto A_{P_{pq}}(x^j, y^{p-j}, x^k, y^{q-k})).$$

Note that the function

$$w_{p,q,j,k}(x) = \varphi(y \mapsto A_{P_{pq}}(x^j, y^{p-j}, x^k, y^{q-k}))$$

is the restriction to the diagonal of (j, k) -linear symmetric mapping

$$B(x_1, \dots, x_j, x_{j+1}, \dots, x_{j+k}) = \varphi(y \mapsto A_{P_{pq}}(x_1, \dots, x_j, y^{p-j}, x_{j+1}, \dots, x_{j+k}, y^{q-k})),$$

therefore, $w_{p,q,j,k}$ is a continuous (j, k) -polynomial. Hence, $h_{\varphi, P}$ is a continuous $*$ -polynomial.

Let us show that $h_{\varphi, f} \in \mathcal{A}(X)$ for every $f \in \mathcal{A}(X)$. Since $f \in \mathcal{A}(X)$ and $\mathcal{P}_*(X)$ is dense in $\mathcal{A}(X)$, there exists the sequence $\{f_n\}_{n=1}^\infty \subset \mathcal{P}_*(X)$, which converges to f . Since $f_n \in \mathcal{P}_*(X)$, it follows that $h_{\varphi, f_n} \in \mathcal{P}_*(X)$. Therefore, the sequence $\{h_{\varphi, f_n}\}_{n=1}^\infty$ is contained in $\mathcal{P}_*(X)$. Let us show that this sequence is fundamental in $\mathcal{A}(X)$. Let $r \in \mathbb{Q}_+$. For $m, n \in \mathbb{N}$, by (8),

$$|h_{\varphi, f_m}(x) - h_{\varphi, f_n}(x)| = |h_{\varphi, f_m - f_n}(x)| \leq K C_s c_{s+\|x\|}^{-1} \|f_m - f_n\|_{s+\|x\|}.$$

Therefore,

$$\|h_{\varphi, f_m} - h_{\varphi, f_n}\|_r \leq C_r \sup_{\|x\| \leq r} |h_{\varphi, f_m}(x) - h_{\varphi, f_n}(x)| \leq K C_r C_s \sup_{\|x\| \leq r} c_{s+\|x\|}^{-1} \|f_m - f_n\|_{s+\|x\|}.$$

Note that

$$\sup_{\|x\| \leq r} c_{s+\|x\|}^{-1} \|f_m - f_n\|_{s+\|x\|} \leq \left(\sup_{\|x\| \leq r} c_{s+\|x\|}^{-1} \right) \left(\sup_{\|x\| \leq r} \|f_m - f_n\|_{s+\|x\|} \right)$$

and

$$\sup_{\|x\| \leq r} c_{s+\|x\|}^{-1} = \left(\inf_{\|x\| \leq r} c_{s+\|x\|} \right)^{-1} = \left(\inf_{t \in [s, s+r]} c_t \right)^{-1},$$

which is finite, because $\inf_{t \in [s, s+r]} c_t > 0$. By (4),

$$\begin{aligned} \sup_{\|x\| \leq r} \|f_m - f_n\|_{s+\|x\|} &\leq \sup_{\|x\| \leq r} C_{s+\|x\|} \sup_{\|y\| \leq s+\|x\|} |f_m(y) - f_n(y)| \leq \\ &\leq \sup_{\|x\| \leq r} C_{s+\|x\|} \sup_{\|y\| \leq s+r} |f_m(y) - f_n(y)| = \left(\sup_{t \in [s, s+r]} C_t \right) \|f_m - f_n\|_{s+r}. \end{aligned}$$

Thus,

$$\|h_{\varphi, f_m} - h_{\varphi, f_n}\|_r \leq KC_r C_s \left(\inf_{t \in [s, s+r]} c_t \right)^{-1} \left(\sup_{t \in [s, s+r]} C_t \right) \|f_m - f_n\|_{s+r}.$$

Therefore, since the sequence $\{f_n\}_{n=1}^{\infty}$ is fundamental, it follows that the sequence $\{h_{\varphi, f_n}\}_{n=1}^{\infty}$ is fundamental. Since $\mathcal{A}(X)$ is complete, there exists $v \in \mathcal{A}(X)$ such that the sequence $\{h_{\varphi, f_n}\}_{n=1}^{\infty}$ converges to v . Let us show that $h_{\varphi, f} = v$. Let $x \in X$. Let $\rho \in \mathbb{Q}_+$ be such that $\rho > \|x\|$. By (4),

$$\sup_{\|z\| \leq \rho} |h_{\varphi, f_n}(z) - v(z)| \leq c_{\rho}^{-1} \|h_{\varphi, f_n} - v\|_{\rho}.$$

Therefore,

$$|h_{\varphi, f_n}(x) - v(x)| \leq c_{\rho}^{-1} \|h_{\varphi, f_n} - v\|_{\rho}.$$

Since $\|h_{\varphi, f_n} - v\|_{\rho} \rightarrow 0$ as $n \rightarrow \infty$, it follows that the sequence $\{h_{\varphi, f_n}(x)\}_{n=1}^{\infty}$ converges to $v(x)$. On the other hand, by the continuity of φ and T_x , since $f_n \rightarrow f$ as $n \rightarrow \infty$, we have $\varphi(T_x(f_n)) \rightarrow \varphi(T_x(f))$ as $n \rightarrow \infty$, i.e., $h_{\varphi, f_n}(x) \rightarrow h_{\varphi, f}(x)$ as $n \rightarrow \infty$. Therefore, $h_{\varphi, f}(x) = v(x)$. Thus, $h_{\varphi, f} = v$ and, consequently, $h_{\varphi, f} \in \mathcal{A}(X)$. \square

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Василишин Т.В. *Деякі властивості операторів зсуву на алгебрах, породжених *-поліномами* // Карпатські матем. публ. — 2018. — Т.10, №1. — С. 206–212.

*-Поліном — це функція на комплексному банаховому просторі X , яка є сумою так званих (p, q) -поліномів. У свою чергу, для невід'ємних чисел p і q , (p, q) -поліном — це функція на просторі X , яка є звуженням на діагональ деякого відображення, визначеного на декартовому степені X^{p+q} , яке є лінійним відносно кожного із своїх перших p аргументів і антилінійним відносно кожного із решти q своїх аргументів. Множина всіх неперервних *-поліномів на просторі X утворює алгебру, яка містить алгебру всіх неперервних поліномів на просторі X як власну підалгебру. Таким чином, поповнення цієї алгебри відносно деяких природних норм є ширшими класами функцій, ніж алгебри аналітичних функцій. З іншого боку, завдяки подібності будови *-поліномів і поліномів, для дослідження таких поповнень можна використовувати техніку, розроблену для дослідження аналітичних функцій на банахових просторах.

У роботі досліджується алгебра Фреше функцій на комплексному банаховому просторі, яка є поповненням алгебри всіх неперервних *-поліномів відносно зліченної системи норм, еквівалентних до норм рівномірної збіжності на замкнених кулях простору. Встановлено деякі властивості оператора зсуву (який діє як додавання деякого фіксованого елемента простору до аргументу функції) на цій алгебрі. Зокрема, показано, що оператори зсуву є добре визначеними неперервними лінійними операторами. Також доведено деякі оцінки для норм значень операторів зсуву. Використовуючи ці результати, досліджено один спеціальний клас функцій із алгебри, який є важливим для опису спектра (множини всіх максимальних ідеалів) алгебри.

Ключові слова і фрази: (p, q) -поліном, *-поліном, оператор зсуву.